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The Geometry of the Space of Knotted Polygons

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The Geometry of the Space of Knotted Polygons

A dissertation submitted in partial satisfaction
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in
Mathematics

by

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June 2018

The Geometry of the Space of Knotted Polygons

To my inspiring and encouraging family.

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Abstract

The Geometry of the Space of Knotted Polygons

by

Kathleen S. Hake

For a positive integer $n \geq 3$, the collection of n-sided polygons embedded in 3-space defines the space of geometric knots. We will consider the subspace of equilateral knots, consisting of embedded n-sided polygons with unit length edges. Paths in this space determine isotopies of polygons, so path-components correspond to equilateral knot types. When $n \leq 5$, the space of equilateral knots is connected. Therefore, we examine the space of equilateral hexagons. Using techniques from symplectic geometry, we can parametrize the space of equilateral hexagons with a set of measure preserving action-angle coordinates. With this coordinate system, we provide new bounds on the knotting probability of equilateral hexagons. Additionally, we study physical properties of equilateral knots, especially thickness.

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Chapter 1

Introduction

Classically a knot can be defined as a closed, non self-intersecting smooth curve embedded in Euclidean 3-space. Two knots are considered to be equivalent if one can be smoothly deformed into another. The question of whether or not two given knots are equivalent proves to be a difficult problem. Much of theory is devoted to developing techniques to answer this question. The study of the invariance of knots has been of interest to not only mathematicians but also biologist, physicists, and computer scientists. Prominent examples of knotting appear in polymers, specifically DNA and proteins. In the early 1970's, it was discovered that enzymes called topoisomerases causes the DNA to change its form. Type II topoisomerases bind to two segments of double-standed DNA, split one of the segments, transport the other through the break, and reseal the break. These studies suggest that the topological configuration, or the knotting, plays a role in understanding the behavior of these enzymes. Sometimes the arbitrary flexibility and lack of thickness in the classical theory of knots does not accurately depict the physical constraints of objects in nature. This inspires questions in the field of physical knot theory and models that seek to capture some of the physical properties.

One question is about the statistical distribution of knot types as a function of the length. For example, what is the probability that an n edged polygon is knotted? The study of knots from a probabilistic viewpoint provides insight into the behavior of typical knots. There are many ways to model random knots. One model is closed self-avoiding walks on the integer cubic lattice. A walk on $\mathbb{Z}^3$ is a sequence $\{X_0, X_1, \cdots, X_n\}$ such that $X_0 = (0, 0, 0)$ and $X_{i+1} - X_i \in \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. If the walk is closed and self-avoiding then $X_n = X_0$ and $X_i \neq X_j$. Connecting the points yields an n-segment polygonal path, representing the knot.

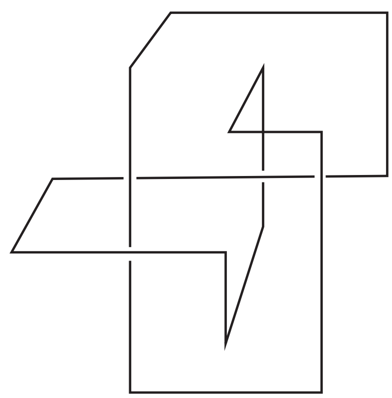


Figure 1.1: The figure shows a trefoil knot on the simple cubic lattice.

A random knot, K_n, is obtaining by sampling a uniform distribution of all closed self-avoiding walks of n steps. Summers and Whittington[1] prove that the probability that K_n is knotted goes to one as n goes to infinity.

A similar model we consider is that of closed polygonal walks in $\mathbb{R}^3$. A knot is realized by joining n line segments. In addition, we restrict the length of the segments to be equal. We identify each n-sided polygonal walk with the $3n$-tuple of vertex coordinates which define it. This gives a correspondence between points in $\mathbb{R}^{3n}$ and n-sided polygons in $\mathbb{R}^3$. We consider the $2(n-3)$ dimensional subspace of equilateral polygons up to translations and rotations. Using techniques from symplectic geometry introduced by Cantarella and

Schonkwiler[2], we study the case of equilateral hexagons. Suppose P is an equilateral hexagon. In Chapter 2, we prove that the probability P is knotted is at most $\frac{14-3\pi}{192} < \frac{1}{42}$.

Other questions motivated by physical restrictions pertain to the thickness or rope-length of a knot. We consider how the polygonal thickness effects the knotting of equilateral hexagons. The optimal conformations for equilateral hexagonal trefoils appear to have certain spatial symmetries. In Chapter 3, we study the space of symmetric equilateral hexagons. In Chapter 4, we discuss the thickness of a polygonal knot and prove the existence of a maximally thick symmetric knotted equilateral hexagon.

1.1 Polygonal Knot Space

There are various ways to define a knot, all of which capture the intuitive notion of a knotted loop. We will start by defining a polygonal knot. For any two distinct points in 3-space, p and q, let $[p, q]$ denote the line segment joining them. For an ordered set of distinct points, $(p_1, p_2, \ldots, p_n)$, the union of the segments $[p_1, p_2], [p_2, p_3], \ldots, [p_{n-1}, p_n]$, and $[p_n, p_1]$ is called a closed polygonal curve. If each segment intersects exactly two other segments, intersecting each only at an endpoint, then the curve is said to be simple.

Definition 1.1.1 *A polygonal knot, K, is a simple, closed polygonal curve in $\mathbb{R}^3$.*

Many choices of ordered sets of points can define the same polygonal knot. For example, cyclicly permuting the order of the points does not change the knot. Also, if three consecutive points are collinear, then eliminating the middle point does not change the knot.

Definition 1.1.2 *If the ordered set $(p_1, p_2, \ldots, p_n)$ defines a polygonal knot, and no proper ordered subset defines the same knot, then the elements of the set $\{p_i\}$ are called vertices of the knot.*

Next we provide a formal definition of deforming knots.

Definition 1.1.3 *A polygonal knot J is called an elementary deformation of the polygonal knot K if one of the two knots is determined by a sequence of points $(p_1, p_2, \ldots, p_n)$ and the other is determined by the sequence $(p_0, p_1, p_2, \ldots, p_n)$, where p_0 is a point which is not collinear with p_1 and p_n and the triangle spanned by (p_0, p_1, p_n) intersects the polygonal knot determined by $(p_1, p_2, \ldots, p_n)$ only in the segment $[p_n, p_1]$.*

Polygonal knots K and J are called equivalent if K can be changed into J by performing a series of elementary deformations.

Definition 1.1.4 *Knots K and J are called equivalent if there is a sequence of knots $K = K_0, K_1, \ldots, K_n = J$, with each K_{i+1} and elementary deformation of K_i, for $i > 0$.*

This notion of equivalence satisfies the definition of an equivalence relation.

If a polygonal knot, P, has n vertices, we will call P an n-sided polygonal knot. We label the vertices of P as $v_1, v_2, \ldots, v_n$. We call the segments $[v_i, v_{i+1}]$ the edges of P and label the edges of P as $e_1, e_2, \ldots, e_n$, where $e_1 = [v_1, v_2], e_2 = [v_2, v_3], \ldots, e_{n-1} = [v_{n-1}, v_n]$, and $e_n = [v_n, v_1]$. In addition, we will select a distinguished vertex, v_1, called a root and a choice of orientation.

With a distinguished vertex and orientation, we can view P as a point in $\mathbb{R}^{3n}$ by listing the coordinates of the vertices starting with v_1 then following the orientation. Not all points in $\mathbb{R}^{3n}$ will correspond to simple polygonal curves. Therefore we define the discriminant set, in the spirit of Vassiliev [3].

Definition 1.1.5 *The discriminant, $\Sigma^{(n)}$, is all points in $\mathbb{R}^{3n}$ that correspond to non-embedded polygonal knots.*

A polygonal knot in $\mathbb{R}^3$ fails to be embedded when two or more of the edges intersect. For an n-sided polygonal knot there are $\frac{1}{2}n(n-3)$ pairs of non-adjacent edges. So $\Sigma^{(n)}$

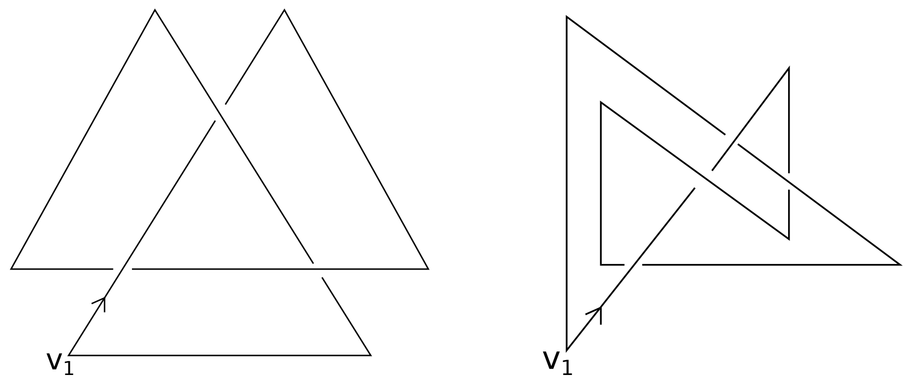


Figure 1.2: The figure on the left shows a 6-sided, rooted, oriented polygonal trefoil knot. The figure on the right shows a 7-sided, rooted, oriented polygonal figure-8 knot.

is the union of the closure of the $\frac{1}{2}n(n-3)$ real semi-algebraic cubic varieties, each piece consisting of polygons with a single intersection between non-adjacent edges[4][5]. For example, the collection of polygons $(v_1, v_2, \ldots, v_n)$ where e_1 and e_3 intersect is the closure of the set of points that satisfy the system

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_4 - v_1) = 0,$$

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1) \times (v_4 - v_1) < 0,$$

$$(v_4 - v_3) \times (v_1 - v_3) \cdot (v_4 - v_3) \times (v_2 - v_3) < 0.$$

The first equation guarantees that v_4 lies in the plane spanned by v_1, v_2, and v_3. If the four vertices are coplanar, the second inequality guarantees that the line containing e_1 separates v_3 and v_4. The third inequality guarantees that the line containing e_3 separates v_1 and v_2. By excluding these singular points, we are left with an open set corresponding to embedded polygons in $\mathbb{R}^n$.

Definition 1.1.6 *The embedding space for rooted, oriented n-sided polygonal knots, denoted $Geo(n)$, is defined to be $\mathbb{R}^{3n} - \Sigma^{(n)}$.*

Then $Geo(n)$ is an open $3n$-dimensional manifold. A continuous path $h : [0,1] \to Geo(n)$ is an isotopy of polygonal knots.

Definition 1.1.7 *Two n-sided polygonal knots are geometrically equivalent if they lie in the same path-component of $Geo(n)$.*

Path components are in bijective correspondence with the geometric knot types realizable with n edges. Any polygon that is in the same path-component of the regular, planar n-gon is then geometrically equivalent to the unknot.

Next we will consider polygonal knots with unit length edges. Consider the function $f : Geo(n) \to \mathbb{R}^n$ where $(v_1, v_2, \ldots, v_n) \mapsto (||v_1 - v_2||, ||v_2 - v_3||, \ldots, ||v_n - v_1||)$.

Definition 1.1.8 *Let $f^{-1}((1,1,\ldots,1)) = Equ(n)$ be the embedding space for rooted, oriented, n-sided equilateral knots.*

Since this space is the preimage of the smooth map f at the regular value $(1,1,\ldots,1)$, $Equ(n)$ is a $2n$-dimensional manifold. Similarly to the space of geometric knots, path-components correspond to the equilateral knot types with n edges. In this dissertation, we will focus on equilateral polygonal knots.

Every triangle is planar. A quadrilateral can be folded along a diagonal to become planar. It is also known that any pentagon can be deformed to a planar pentagon [5]. Let $P \in Equ(5)$ and rotate P so that vertices v_1, v_2, and v_3 are on the xy-plane. If v_4 also lies on the xy-plane, then P can be deformed into a quadrilateral. If not, then edge e_4 intersects the xy-plane. First consider the case where e_4 pierces the interior of the triangular disk spanned by (v_1, v_2, v_3), shown in Figure 1.3. Then the intersection of P with the triangular disk spanned by (v_5, v_1, v_2) is trivial. Therefore P can be deformed into a quadrilateral with vertices (v_2, v_3, v_4, v_5). If e_4 does not pierce the interior of the triangular disk spanned by (v_1, v_2, v_3), then P can be deformed into a quadrilateral with

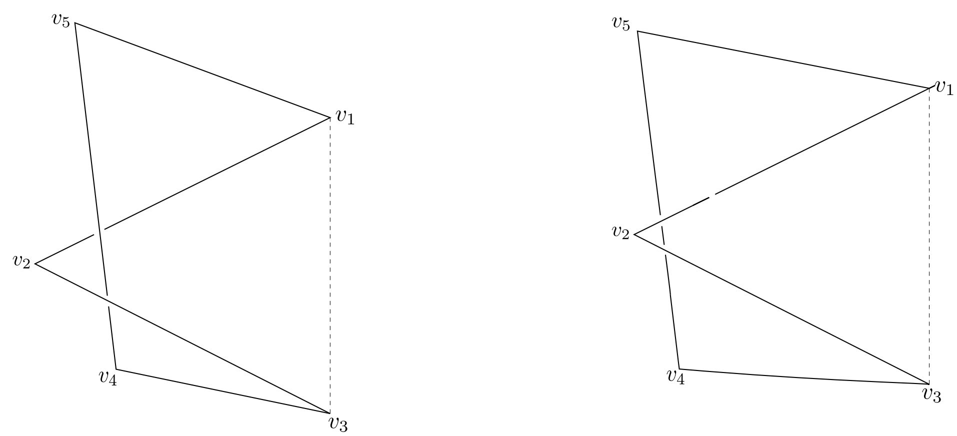


Figure 1.3: The figure on the left shows the case where e_4 pierces the triangular disk spanned by (v_1, v_2, v_3). On the right, e_4 does not pierce the triangular disk spanned by (v_1, v_2, v_3).

vertices (v_1, v_3, v_4, v_5). Since all quadrilaterals are geometrically equivalent to the unknot, then all pentagons are also geometrically equivalent to the unknot. Therefore $Equ(n)$ is connected for $n \leq 5$ and the case of hexagons is the first interesting example. Jorge Calvo [6] proves that $Equ(6)$ has five path-components. One component corresponding to the unknot, two for the right-handed trefoil and two for the left-handed trefoil. In order to distinguish between the different components, he introduces new knot invariants for equilateral hexagonal knots. First let $H = (v_1, v_2, \ldots, v_6) \in Equ(6)$.

Definition 1.1.9 *Let $H \in Equ(6)$. The curl of H, denoted $curl(H)$, is defined by $curl(H) = sign((v_3 - v_1) \times (v_5 - v_1) \cdot (v_2 - v_1))$.*

If v_1, v_3 and v_5 are on the xy-plane oriented in a counter-clockwise orientation, then $curl(H)$ denotes the sign of the z-coordinate of v_2. So, in a sense, some trefoils curl up while others curl down. We will describe a second invariant of the hexagonal knot that distinguishes its topological knot type.

Definition 1.1.10 *Define T_i to be the interior of the triangular disk spanned by (v_{i-1}, v_i, v_{i+1}).*

Using a right-hand rule, we orient each T_i as shown in Figure 1.4.

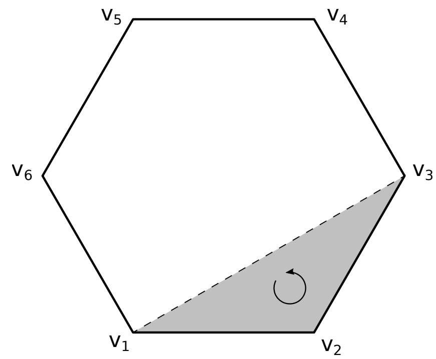


Figure 1.4: In this figure triangular disk T_2 is shaded and the orientation from the right-hand rule is shown.

Definition 1.1.11 *Define Δ_i, for $i = 2, 4,$ and 6, to be the algebraic intersection number of T_i with H.*

First we will consider Δ_2. If no edges of H intersect T_2, then clearly $\Delta_2 = 0$. The only edges that can intersect T_2 are e_4 and e_5. If only one of e_4 or e_5 pierce T_2, then $\Delta_2 = \pm 1$, depending on whether the orientation of the edge agrees with the orientation on T_2. Lastly, if both edges intersect T_2, then $\Delta_2 = 0$ for the edges intersect in opposite directions. So $\Delta_2 \in \{-1, 0, 1\}$.

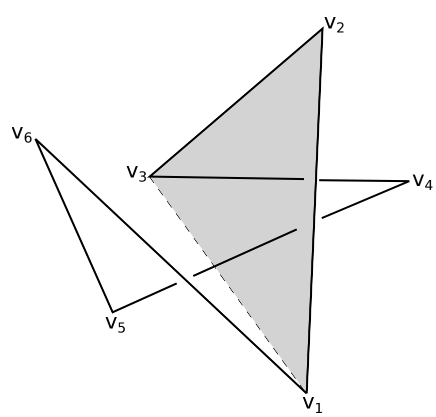


Figure 1.5: This figure shows a hexagonal knot in which $\Delta_2 = 1$.

The only edges of H that can intersect T_4 are e_1 and e_2. Either one, both, or neither will intersect T_4 and so $\Delta_4 \in \{-1, 0, 1\}$. Similarly $\Delta_6 \in \{-1, 0, 1\}$. The following Lemma distinguishes topological knot type using the algebraic intersection numbers.

Lemma 1.1.12 *[6] Let $H \in Equ(6)$. Then*

1. *H is right-handed trefoil iff $\Delta_i = 1$ for all i,*
2. *H is left-handed trefoil iff $\Delta_i = -1$ for all i,*
3. *H is unknot iff $\Delta_i = 0$ for some $i \in \{2, 4, 6\}$.*

Discuss the proof of this lemma and how there really are no other cases, namely if $\Delta_2 = -1$ and $\Delta_4 = -1$ then it is required the $\Delta_6 = 0$.

Combining the notion of curl with the appropriate intersections from Lemma 1.1.12 we arrive at Calvo's Geometric Knot Invariant, Joint Chirality-Curl.

Definition 1.1.13 *[6] Let $H \in Equ(6)$. Define Joint Chirality-Curl*

$$J(H) = (\Delta_2\Delta_4\Delta_6, \Delta_2^2\Delta_4^2\Delta_6^2 curl(H)).$$

The Joint Chirality-Curl distinguishes between the five components of $Equ(6)$.

Theorem 1.1.14 *[6]: Let $H \in Equ(6)$. Then*

$$J(H) = \begin{cases} (0,0) & \text{iff } H \text{ is unknot} \\ (1,c) & \text{iff } H \text{ is right-trefoil with curl(H)=c} \\ (-1,c) & \text{iff } H \text{ is left-trefoil with curl(H)=c} \end{cases}$$

The five components of $Equ(6)$ are due to the choice of a root and orientation. Consider the automorphisms r and s on $Equ(6)$ defined by

$$r\langle v_1, v_2, v_3, v_4, v_5, v_6\rangle = \langle v_1, v_6, v_5, v_4, v_3, v_2\rangle$$

$$s\langle v_1, v_2, v_3, v_4, v_5, v_6\rangle = \langle v_2, v_3, v_4, v_5, v_6, v_1\rangle.$$

These automorphisms act on $Equ(6)$ by reversing or shifting the order of the vertices of each hexagon. They generate the dihedral group of order twelve.

Theorem 1.1.15 *[6] Suppose Γ is a subgroup of the dihedral group $\langle r, s\rangle$. Then $Geo(6)/\Gamma$ has five components if and only if Γ is contained in the index-2 subgroup $\langle s^2, rs\rangle$. Otherwise, $Geo(6)/\Gamma$ has three components.*

Corollary 1.1.16 *[6] The spaces $Geo(6)/\langle s\rangle$ of non-rooted oriented embedded hexagons, and $Geo(6)/\langle r, s\rangle$ of non-rooted non-oriented embedded hexagons, each consist of three path-components.*

1.2 Symplectic Geometry

Next we will discuss definitions and results from symplectic geometry, specifically toric symplectic manifolds[7], that apply to knot spaces.

Definition 1.2.1 *A symplectic manifold, M, is an even dimensional manifold with a closed, non-degenerate 2-form, ω, called the symplectic form.*

A useful example to consider is the unit sphere S^2, where ω is the standard area form. If $S^2 = \{(x_1, x_2, x_3) : \sum_j x_j^2 = 1\}$, then $\omega_x(u, v) = \langle x, u \times v\rangle$ for $u, v \in T_xS^2$.

Since ω is nondegenerate there is a canonical isomorphism between the tangent and cotangent bundles, namely

$$TM \mapsto T^*M : X \to \iota(X)\omega = \omega(X, \cdot).$$

Definition 1.2.2 *A symplectomorphism of a symplectic manifold (M, ω) is a diffeomorphism $\psi \in Diff(M)$ that preserves the symplectic form. The group of symplectomorphisms of M is denoted $Symp(M, \omega)$.*

Since ω is nondegenerate the homomorphism $T_qM \to T_q^*M : v \mapsto \iota(v)\omega_q$ is bijective. Thus there is a one-to-one correspondence between vector fields and 1-forms via

$$\chi(M) \to \Omega^1(M) : X \mapsto \iota(X)\omega.$$

Definition 1.2.3 *A vector field $X \in \chi(M)$ is called symplectic if $\iota(X)\omega$ is closed. Denote the space of symplectic vector fields by $\chi(M, \omega)$.*

Proposition 1.2.4 *[7] Let M be a closed manifold. If $t \mapsto \psi_t \in Diff(M)$ is a smooth family of diffeomorphims generated by a family of vector fields $X_t \in \chi(M)$ via*

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = id,$$

then $\psi_t \in Symp(M, \omega)$ for every t if and only if $X_t \in \chi(M, \omega)$ for every t.

Now consider a smooth function $H : M \to \mathbb{R}$.

Definition 1.2.5 *The vector field $X_H : M \to TM$ determined by identity $dH = \iota(X_H)\omega$ is called the Hamiltonian vector field associated to H.*

If M is closed, then by Proposition 1.2.4 the vector field X_H generates a smooth 1-parameter group of diffeomorphisms $\phi_H^t \in Diff(M)$ such that

$$\frac{d}{dt}\phi_H^t = X_H \circ \phi_H^t, \qquad \phi_H^0 = id,$$

called a Hamiltonian flow associated to H. The identity

$$dH(X_H) = (\iota(X_H)\omega)(X_H) = \omega(X_H, X_H) = 0$$

shows that X_H is tangent to level sets.

Let us return to our example of the unit 2-sphere. Consider cylindrical polar coordinates (θ, x_3) for $\theta \in [0, 2\pi)$ and $x_3 \in [-1, 1]$. Let H be the height function x_3 on S^2. The level sets are circles at constant height. The Hamiltonian flow ϕ_H^t rotates each circle at constant speed and X_H is the vector field $\frac{\partial}{\partial\theta}$. Thus $\phi^t(H)$ is the rotation of the sphere about its vertical axis through the angle t.

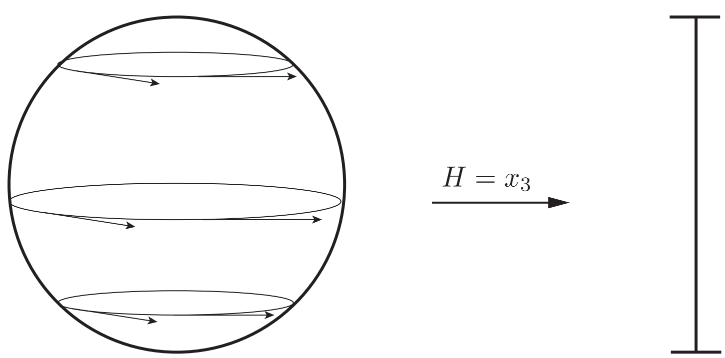


Figure 1.6: Rotating the sphere about its vertical axis.

Consider a smooth map $[0,1] \times M \to M : (t,q) \mapsto \psi_t(q)$ such that $\psi_t \in Symp(M,\omega)$ and $\psi_0 = id$. A family of such symplectomorphisms is called a symplectic isotopy of M. The isotopy is generated by a unique family of vector fields $X_t : M \to TM$ such that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$. If all of the 1-forms are exact then there exists a smooth family of Hamiltonian functions $H_t : M \to \mathbb{R}$ such that $\iota(X_t)\omega = dH_t$. In this case, ψ_t is called a Hamiltonian isotopy.

Definition 1.2.6 *A symplectomorphism, ψ, is called Hamiltonian if there exists a Hamiltonian isotopy $\psi_t \in Symp(M,\omega)$ from $\psi_0 = id$ to $\psi_1 = \psi$.*

Definition 1.2.7 *A Hamiltonian action of S^1 on (M,ω) is a 1-parameter subgroup $\mathbb{R} \to Symp(M) : t \mapsto \psi_t$ of $Symp(M)$ where $\psi_t = id$ and which is the integral of a Hamiltonian vector field X_H.*

The Hamiltonian function $H : M \to \mathbb{R}$ in this case is called the moment map. If k such symmetries commute we have an action of a torus, T^k, on M. Then the moment map, $\mu : M \to \mathbb{R}^k$ yields a k-dimensional vector of conserved quantities. If k is half the dimension of M, then M is called toric symplectic. From theorems of Atiyah[8] and Guillemin-Sternberg[9], the image of μ is a convex polytope, P, called the moment polytope. Moreover, the vertices of the moment polytope are the images under μ of the fixed point of the Hamiltonian torus action. In addition, the torus action preserves the fibers of the moment map. If we can invert μ, we get a map $\alpha : P \times T^n \to M$ called the action-angle map.

The example of the unit sphere S^2 is a toric symplectic manifold, with circle action rotation about the z-axis. The moment map $H : S^2 \to S^1$ is the height function, the conserved quantity as the sphere rotates. The image of H is a convex polytope, namely the interval $[-1,1]$. The fibers of H are horizontal circles of constant height, which are preserved under the action. Lastly the circle, S^1, is half the dimension of S^2.

The toric symplectic structure on the sphere naturally carries over to a toric symplectic structure on the product of spheres. This gives a toric symplectic structure on the space of open random walks or open polygons. We will consider the subspace of closed random walks. Let $Pol(n)$ be the $2n$ dimensional space of possibly singular polygons in $\mathbb{R}^3$ with edgelengths one. We will consider the quotient space $Pol_0(n) = Pol(n)/\mathrm{SO}(3)$ of equilateral polygons up to translations and rotations. Jason Cantarella and Clayton Shonkwiler [2] describe the almost toric symplectic structure of $Pol_0(n)$. To define the toric action, consider any triangulation, T, of an equilateral planar regular n-gon. Let d_i be the lengths of the $n-3$ diagonals of the triangulation. These diagonals along with the edges on the polygon form $n-2$ triangles which each obey 3 triangle inequalities. Therefore the lengths of the diagonals and the edge lengths must obey a set of $3(n-2)$ triangle inequalities, called the triangulation inequalities.

Theorem 1.2.8 *[10][11][12] The following facts are known:*

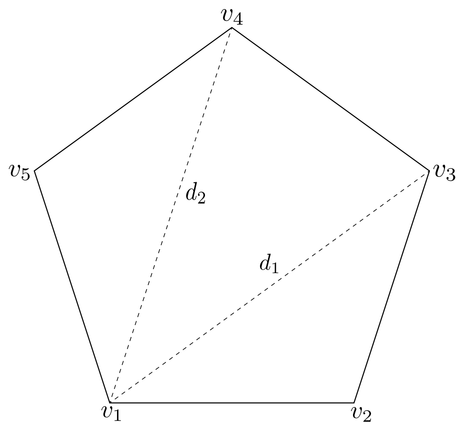
- *$Pol_0(n)$ is a possibly singular $(2n-6)$-dimensional symplectic manifold. The symplectic volume is equal to the standard measure.*

- *To any triangulation T of the standard n-gon we can associate a Hamiltonian action of the torus T^{n-3} on $Pol_0(n)$, where θ_i acts by folding the polygon around the ith diagonal of the triangulation.*

- *The moment map $\mu : Pol_0(n) \mapsto \mathbb{R}^{n-3}$ for a triangulation T records the lengths d_i of the $n-3$ diagonals of the triangulation.*

- *The inverse image $\mu^{-1}(int(P)) \subset Pol_0(n)$ of the interior of the moment polytope P is an toric symplectic manifold.*

The moment polytope, P_n, is defined by the triangulation inequalities for T. The vertices of the moment polytope represent degenerate polygons which extremize several triangulation inequalities. Figure 1.7 shows a triangulation of an equilateral pentagon and the corresponding moment polytope.



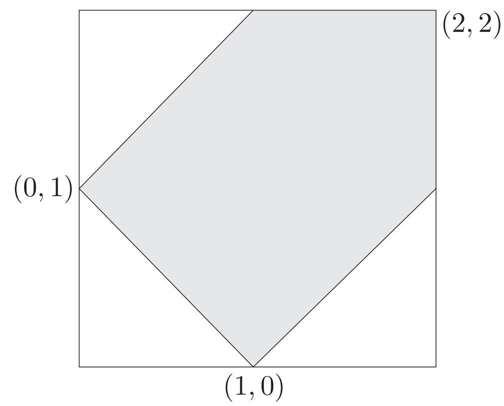


Figure 1.7: The left image shows the fan triangulation of an equilateral pentagon, where all diagonals share a common vertex. The lengths of the diagonals, d_1 and d_2, satisfy six triangle inequalities. The figure on the left shows the moment polytope of $Pol_0(5)$ corresponding to the fan triangulation.

The action-angle map $\alpha : P_n \times T^{n-3} \mapsto Pol_0(n)$ for a triangulation T is given by first constructing the $n-2$ triangles using the diagonal lengths, d_i, and edge llengths of 1 and then joining them in 3-space with dihedral angles given by the θ_i. The polygon is the boundary of this triangulated surface. This construction only makes sense for polygons equivalent up to translations and rotations, which is why the quotient by SO(3) is necessary. An example of an equilateral pentagon is shown in Figure 1.8.

The following theorem will be used in Chapter 3 when calculating the knotting probability of equilateral hexagons.

Theorem 1.2.9 *(Duistermaat-Heckman)[13] Suppose M is a $2n$-dimensional toric symplectic manifold with moment polytope P, T^n is the n-torus and α inverts the moment map. If we take the standard measure on the n-torus and the uniform measure on int(P), then the map $\alpha : int(P) \times T^n \to M$ parametrizing a full-measure subset of M in action-angles coordinates is measure-preserving. In particular, if $f : M \to \mathbb{R}$ is any integrable*

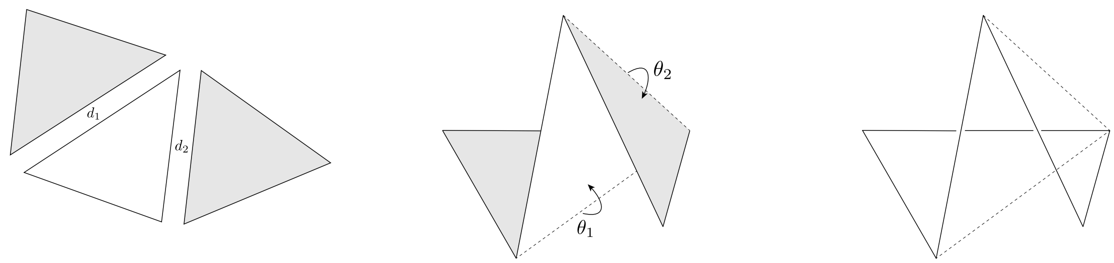


Figure 1.8: The figure shows how to construct a equilateral pentagon from the action-angle map $\alpha : P_5 \times T^2 \mapsto Pol_0(5)$ for the fan triangulation. A point (d_1, d_2) in the moment polytope gives the information needed to construct three triangles. Then a point $(\theta_1, \theta_2) \in T^2$ gives instruction on how to attach the triangles along the diagonals. The boundary of the triangulated surface is the equilateral pentagon.

function then

$$\int_M f(x)\ dm = \int_{P\times T^n} f(d_1, \cdots, d_n, \theta_1, \cdots, \theta_n)\ dVol_{\mathbb{R}^n} \wedge d\theta_1 \wedge \cdots \wedge d\theta_n$$

and if $f(d_1, \cdots, d_n, \theta_1, \cdots, \theta_n) = f_d(d_1, \cdots, d_n) f_\theta(\theta_1, \cdots, \theta_n)$ *then*

$$\int_M f(x)\ dm = \int_P f_d(d_1, \cdots, d_n) dVol_{\mathbb{R}^n} \int_{T^n} f_\theta(\theta_1, \cdots, \theta_n) d\theta_1 \wedge \cdots \wedge d\theta_n.$$

Chapter 2

Symplectic Structure of the Space of Equilateral Hexagons

In this Chapter, we will discuss the almost toric symplectic structure on the space of equilateral hexagons. The toric symplectic structure gives a set of measure preserving action-angle coordinates to parametrize the embedding space of rooted, oriented equilateral hexagonal knots. In sections 3.2, 3.3, 3.4, and 3.5 constraints on which action-angle coordinate correspond to which geometric knot type are given. These results are used to show that the probability than an equilateral hexagon is knotted is at most $\frac{14-2\pi}{192} < \frac{1}{42}$.

2.1 Equilateral Hexagons

In order to describe the action-angle coordinates on the space of equilateral hexagonal knots, we first must consider the quotient space of $Equ(6)$ by $SO(3)$.

Definition 2.1.1 *Let $Equ_0(6) = Equ(6)/SO(3) \times \mathbb{R}$ be the embedding space of rooted, oriented equilateral hexagonal knots up to translations and rotations.*

Let $H = (v_1, v_2, v_3, v_4, v_5, v_6) \in Equ(6)$. We can translate H so that $v_1 = (0, 0, 0)$.

Additionally we rotate H so that v_3 is on the positive x-axis and v_5 on the upper-half xy-plane. Therefore v_1, v_3, and v_5 are on the xy-plane in a counter-clockwise orientation. In this chapter, we will consider this to be the standard position for $H \in Equ_0(6)$.

Next we can choose any triangulation of the standard planar equilateral hexagon to form our action-angle coordinates. We will use one of the triangulations that has a central triangle.

Definition 2.1.2 *Let the T_{135} triangulation be the triangulation of the regular planar equilateral hexagon which has diagonals connecting v_1 to v_3, v_3 to v_5, and v_5 to v_1, with lengths d_1, d_2, and d_3, respectively.*

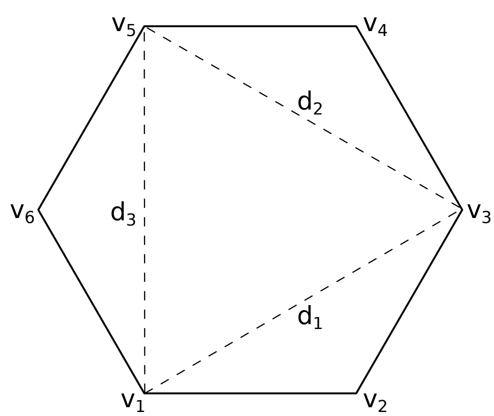


Figure 2.1: This figure shows the T_{135} triangulation of an equilateral hexagon.

The lengths of the diagonals of the T_{135} triangulation obey the following triangulation inequalities:

$$\begin{aligned} 0 \leq d_1 &\leq 2, & & d_3 \leq d_1 + d_2, \\ 0 \leq d_2 &\leq 2, & \text{and} \quad & d_1 \leq d_3 + d_2, \\ 0 \leq d_3 &\leq 2, & & d_2 \leq d_3 + d_1. \end{aligned}$$

Definition 2.1.3 *The T_{135} triangulation polytope, P_6, is the moment polytope for $Pol_0(6)$ corresponding to the T_{135} triangulation and is determined by the triangulation inequalities.*

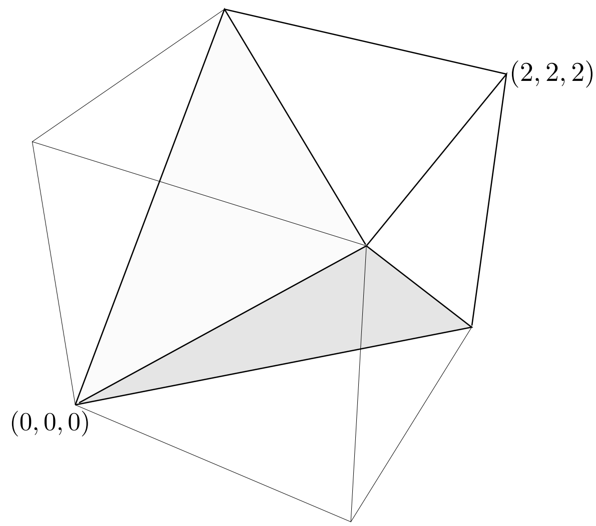


Figure 2.2: This figure shows the T_{135} triangulation polytope.

Let θ_i be the dihedral angle around diagonal d_i, where the regular planar hexagon has all angles π. Then the action-angle map for T_{135}, $\alpha : P_6 \times T^3 \mapsto Pol_0(6)$ allows us to parametrize any $H \in Equ_0(6)$ as $H = (d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$. To construct an equilateral hexagonal knot in $Equ_0(6)$, first choose a point $(d_1, d_2, d_3) \in P_6$ and construct four triangles: one with lengths d_1, d_2, and d_3 and three isosceles triangles with two side lengths 1 and third side d_i. The triangle with side lengths d_1, d_2, and d_3 is placed on the xy-plane with v_1 the origin and v_3 on the positive x-axis. Then a point $(\theta_1, \theta_2, \theta_3)$ in the torus T^3 gives instructions on how to connect the three remaining triangles.

For $H \in Equ_0(6)$ in standard position, the action-angle coordinates arising from the T_{135} triangulation gives the following coordinates for the vertices of H:

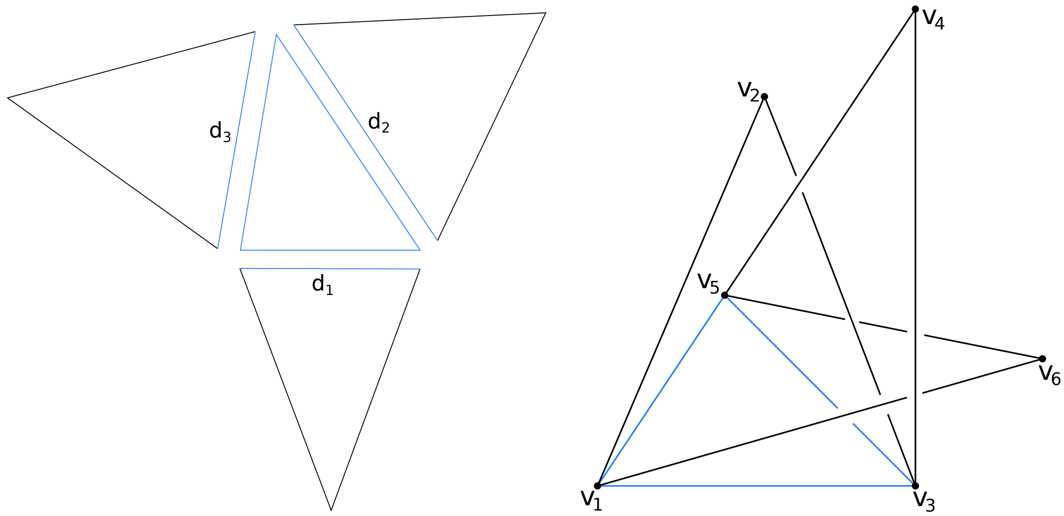


Figure 2.3: Given a point $(d_1, d_2, d_3) \in P_6$ four triangles are formed. Then given a triple of angles, the triangles are connected to form an equilateral hexagonal trefoil.

$$
\begin{aligned}
v_1 =& \Big(0, 0, 0\Big) \\
v_2 =& \Big(\frac{d_1}{2}, \frac{1}{2}\sqrt{4-(d_1)^2}\,\cos(\theta_1), \frac{1}{2}\sqrt{4-(d_1)^2}\,\sin(\theta_1)\Big), \\
v_3 =& \Big(d_1, 0, 0\Big), \\
v_4 =& \Big(\frac{3(d_1)^2-(d_2)^2+(d_3)^2}{4d_1} - \frac{d}{4d_1d_2}\sqrt{4-(d_2)^2}\,\cos(\theta_2), \frac{d}{4d_1} - \\
& \frac{(d_1)^2+(d_2)^2-(d_3)^2}{4d_1d_2}\sqrt{4-(d_2)^2}\,\cos(\theta_2), \frac{1}{2}\sqrt{4-(d_2)^2}\,\sin(\theta_2)\Big), \\
v_5 =& \Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1}, \frac{d}{2d_1}, 0\Big), \\
v_6 =& \Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{4d_1} - \frac{d}{4d_1d_3}\sqrt{4-(d_3)^2}\,\cos(\theta_3), \frac{d}{4d_1} - \\
& \frac{(d_1)^2-(d_2)^2+(d_3)^2}{4d_1d_3}\sqrt{4-(d_3)^2}\,\cos(\theta_3), \frac{1}{2}\sqrt{4-(d_3)^2}\,\sin(\theta_3)\Big),
\end{aligned}
$$

where $d = \sqrt{2(d_1d_2)^2 + 2(d_1d_3)^2 + 2(d_2d_3)^2 - (d_1)^4 - (d_2)^4 - (d_3)^4}$.

Recall that the geometric knot invariant for hexagons, Joint Chirality-Curl, distinguishes between two types of both right-handed and left-handed trefoils with $curl(H) =$

$\mathrm{sign}((v_3 - v_1) \times (v_5 - v_1) \cdot (v_2 - v_1))$. The following two lemmas give a relation between the curl of a hexagon and the possible dihedral angles.

Lemma 2.1.4 *Let $H \in Equ_0(6)$. Let H be parametrized using action-angle coordinates from the T_{135} triangulation. If H has Joint Chirality-Curl $(\pm 1, 1)$, then $\theta_i \in (0, \pi)$ for $i = 1, 2, 3$.*

Proof: Let $H \in Equ_0(6)$ be parametrized with action-angle coordinates $(d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ arising from the T_{135} triangulation. Let H be in standard position. Since v_1, v_3, and v_5 are on the xy-plane oriented in a counter-clockwise direction, $curl(H)$ denotes the sign of the z-coordinate of v_2. Therefore if $curl(H) = 1$, then $\theta_1 \in (0, \pi)$. Suppose that $\theta_2, \theta_3 \in (\pi, 2\pi)$. Then both e_4 and e_5 lie below the xy-plane and can not pierce T_2. Thus $\Delta_2 = 0$ and H has Joint Chirality-Curl $(0, 0)$. If $\theta_2 \in (\pi, 2\pi)$ and $\theta_3 \in (0, \pi)$, then neither e_6 nor e_1 can pierce T_4 and $\Delta_4 = 0$. Similarly if $\theta_3 \in (\pi, 2\pi)$ and $\theta_2 \in (0, \pi)$, then $\Delta_6 = 0$. Therefore if $curl(H) = 1$, then $\theta_i \in (0, \pi)$ for all $i \in 1, 2, 3$. $\square$

Lemma 2.1.5 *Let $H \in Equ_0(6)$. Let H be parametrized using action-angle coordinates from the T_{135} triangulation. If H has Joint Chirality-Curl $(\pm 1, -1)$, then $\theta_i \in (\pi, 2\pi)$ for $i = 1, 2, 3$.*

Proof: Let $H \in Equ_0(6)$ be parametrized with action-angle coordinates $(d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ arising from the T_{135} triangulation. Let H be in standard position. If $curl(H) = -1$, then $\theta_1 \in (\pi, 2\pi)$. Similar to the previous argument of Lemma 2.1.4, if either of both of θ_2 and θ_3 are between 0 and π, then $J(H) = (0, 0)$. Therefore if $curl(H) = -1$ and H is a trefoil, then $\theta_i \in (\pi, 2\pi)$ for all $i \in 1, 2, 3$. $\square$

2.2 Equilateral, Right-Handed, Positive Curl, Hexagonal Trefoils

In this section, we will determine the constraints on the values of $d_1, d_2, d_3, \theta_1, \theta_2$, and θ_3 in order to have a right-handed trefoil with positive curl. First we will define a set of inequalities that must be satisfied in order for $H \in Equ_0(6)$ in standard position to have Joint Chirality-Curl $(1, 1)$.

Proposition 2.2.1 *Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates arising from the T_{135} triangulation. If $J(H) = (1, 1)$, then the following nine functions must be positive:*

$$
\begin{aligned}
f_1 =& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_3 d - ((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big)- \\
& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_2 d - ((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big), \\
g_1 =& \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_2)sin(\theta_3)+sin(\theta_2)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3}, \\
h_1 =& \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_3)sin(\theta_2)+sin(\theta_3)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2}, \\
f_2 =& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_1 d - ((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big)- \\
& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_3 d - (-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big), \\
g_2 =& \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_3)sin(\theta_1)+sin(\theta_3)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1}, \\
h_2 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_1)sin(\theta_3)+sin(\theta_1)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3}, \\
f_3 =& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_2 d - (-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big)- \\
& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_1 d - ((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big), \\
g_3 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_1)sin(\theta_2)+sin(\theta_1)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2}, \\
h_3 =& \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_2)sin(\theta_1)+sin(\theta_2)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1}.
\end{aligned}
$$

Proof: Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the T_{135} triangulation. If $curl(H) = 1$, then $\theta_i \in (0,\pi)$ for all i by Lemma 2.1.4. Recall that for a hexagon, $H \in Equ_0(6)$ to be a right-handed trefoil then the algebraic intersection numbers Δ_i have to be equal to one for $i = 2,4,6$. First we will consider $\Delta_4 = 1$. This means that the triangular disk T_4 containing v_3, v_4, and v_5 must be pierced by either the edge e_6 or e_1 so that the orientation on the edge agrees with the orientation on T_4 coming from a right-hand rule. If $\theta_2 \in (0,\pi)$, then e_6 must pierce T_4 for $\Delta_4 = 1$.

In order for the line going through v_1 and v_6 to pierce T_4 the following must be positive:

$$(v_6 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) > 0, \tag{2.1}$$

$$(v_6 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) > 0, \tag{2.2}$$

$$(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) > 0. \tag{2.3}$$

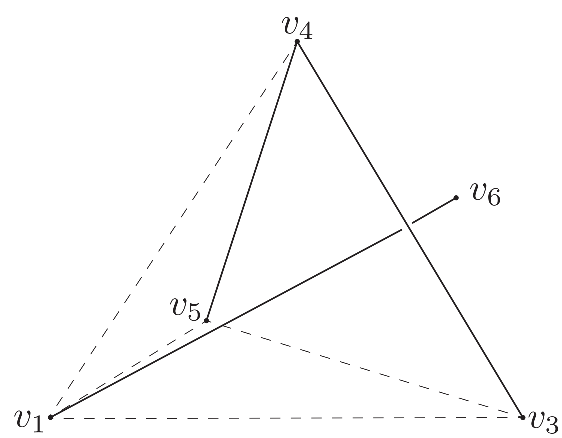


Figure 2.4: This figure shows the case where e_6 pierces T_6.

In addition, the plane containing v_3, v_4, and v_5 must separate v_1 from v_6. So the following must be negative

$$\big((v_6 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big)\big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) < 0. \tag{2.4}$$

Since $\theta_i \in (0, \pi)$ for all i, $(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1)$ is always positive. Negating (2.4) leaves three inequalities that must be satisfied so that e_6 pierces T_4. Letting H be in standard position and evaluating the remaining three with action-angle coordinates gives three functions f_1, g_1, h_1 that must be positive for H to have Joint Chirality-Curl $(1, 1)$.

Next it is required that $\Delta_6 = 1$ for H to have Joint Chirality-Curl $(1, 1)$. Therefore either e_2 or e_3 pierce T_6. If H is in standard position, then e_2 must piece T_6 for $\Delta_6 = 1$. For the line through v_1 and v_2 to go through the interior of T_6 the following three equations must be positive:

$$(v_2 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) > 0, \tag{2.5}$$

$$(v_2 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) > 0, \tag{2.6}$$

$$(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0. \tag{2.7}$$

In addition, the plane containing v_5, v_6, and v_1 must separate v_2 and v_3. Therefore the following equation must be negative:

$$\big((v_2 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big)\big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) < 0. \tag{2.8}$$

Similar to the previous case, 2.7 is always positive if $\theta_i \in (0, \pi)$. The remaining three inequalities reduce to f_2, g_2, h_2 being positive.

Lastly, for $J(H) = (1, 1)$, it is required that $\Delta_2 = 1$. This means that either e_4 or e_5 pierce T_2. If $\theta_1 \in (0, \pi)$, then e_4 must pierce T_2 for $\Delta_2 = 1$. In order for the line passing through v_4 and v_5 to go through the interior of the triangular disk T_2, the three equations must be positive:

$$(v_4 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) > 0, \tag{2.9}$$

$$(v_4 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) > 0, \tag{2.10}$$

$$(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0. \tag{2.11}$$

In addition, the plane containing v_1, v_2, and v_3 must separate v_4 from v_5. Thus the following equation must be negative:

$$\big((v_4 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big)\big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) < 0. \tag{2.12}$$

Similar to the previous case, by evaluating with action-angle coordinates we see that (2.11) is always positive. That leaves three functions f_3, g_3, and h_3 that must be positive so that $\Delta_2 = 1$.

Therefore if H is in standard position and parametrized with action-angle coordinates form the T_{135} triangulation, f_i, g_i, and h_i must be positive for H to have Joint Chirality-Curl $(1, 1)$. □

Next we will prove constraints on action-angle coordinates to get a right-handed positive curl trefoil.

Lemma 2.2.2 *Let $H \in Equ_0(6)$ and parametrize H with with action-angle coordinates coming from the T_{135} triangulation. If H has Joint Chirality-Curl $(1, 1)$, then the lengths of diagonals d_i must be distinct.*

Proof: Let H be in $Equ_0(6)$. Parametrize H with action-angle coordinates coming from the T_{135} triangulation, so $H = (d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ and H is in standard position. Suppose $d_1 = d_2 = d_3 = x$, for some $x \in (0,2)$. Then $v_1 = (0,0,0), v_3 = (x,0,0)$, and $v_5 = (\frac{x}{2}, \frac{\sqrt{3}x}{2}, 0)$. First we will consider the case when $x = \sqrt{3}$. When $\theta_i = 0$ for all i, H is planar but singular with vertices v_2, v_4, and v_6 coinciding in a single point, $(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$. Additionally $e_1 = e_6$, $e_2 = e_3$, and $e_4 = e_5$. As θ_1 increases from 0 to π, v_2 traverses a circle, c_2, of radius $\frac{1}{2}$ centered at $(\frac{\sqrt{3}}{2}, 0, 0)$ lying in a plane parallel to the yz-plane. Similarly, as θ_3 increases from 0 to 2π, v_6 moves along a circle, c_6, of radius $\frac{1}{2}$ centered at the midpoint of the edge connecting v_1 and v_5. Therefore e_1 sweeps out a circular cone, C_{12}, with vertex the origin and base circle c_2 and e_6 sweeps out a circular cone, C_{61}, with vertex the origin and base circle c_6. The circles c_2 and c_6 only intersect when $\theta_1 = \theta_3 = 0$. Therefore the respective cones only intersect in the segment from $(0,0,0)$ to $(\frac{\sqrt{3}}{2}, 0, 0)$, corresponding to edges e_1 and e_6 coinciding when $\theta_1 = \theta_3 = 0$. This implies that e_2 can not pierce T_6. Thus if $x = \sqrt{3}$, H can not have Joint Chirality-Curl $(1,1)$.

Next we consider the case when $\sqrt{3} < x < 2$. When $\theta_1, \theta_2, \theta_3 = 0$, H is planar and embedded. Hence H is unknotted. Cones C_{12} and C_{61}, formed by edges e_1 and e_6 as θ_1 and θ_3 vary, do not intersect. Therefore neither T_2 nor T_6 will be pierced by H, so H will remain unknotted.

Now consider the case when $0 < x < \sqrt{3}$. If $\theta_i = \cos^{-1}(\frac{\sqrt{3}x}{3\sqrt{4-x^2}})$ for all i, then v_2, v_4, and v_6 coincide. If $\theta_i \in (\cos^{-1}(\frac{\sqrt{3}x}{3\sqrt{4-x^2}}), \pi)$ for any $i \in \{1,2,3\}$ then H will be unknotted. Therefore suppose that $\theta_i \in (0, \cos^{-1}(\frac{\sqrt{3}x}{3\sqrt{4-x^2}}))$ for all i. If $\theta_1 = \theta_3 = 0$, then e_2 and e_5 intersect in a point on the xy-plane. If $1 \leq x < \sqrt{3}$, then the point of intersection is interior of the triangle with vertices v_1, v_3, and v_5. As θ_1 and θ_3 increase from 0 to $\cos^{-1}(\frac{\sqrt{3}x}{3\sqrt{4-x^2}})$, e_2 and e_5 continue to intersect in a point. In order for e_2 to pierce T_6, then $\theta_3 > \theta_1$. Similarly e_1 and e_4 intersect when $\theta_1 = \theta_2$. In order for e_4 to pierce T_2, then $\theta_1 > \theta_2$. When $\theta_2 = \theta_3$, e_6 and e_3 intersect. In order for e_6 to pierce T_4, then $\theta_2 > \theta_3$. This implies that $\theta_3 > \theta_1 > \theta_2 > \theta_3$, a contradiction. When $0 < x < 1$ and $\theta_1 = \theta_3 = 0$, then e_2 and e_5 intersect in a point that is exterior of the triangle with vertices v_1, v_3,

and v_5. In order for e_2 to pierce T_6, then $\theta_3 < \theta_1$. In order for e_4 to pierce T_2, then $\theta_1 < \theta_2$. In order for e_6 to pierce T_4, then $\theta_2 < \theta_3$. This implies that $\theta_3 < \theta_1 < \theta_2 < \theta_3$, a contradiction. Therefore when $d_1 = d_2 = d_3$, H can not have Joint Chirality-Curl $(1, 1)$.
$\square$

From Lemma 2.1.4, we know that all three dihedral angles must be in the interval $(0, \pi)$ for $curl(H) = 1$. Next given any admissible triple of diagonal lengths, we prove a tighter constraints on the dihedral angles so that H has Joint Chirality-Curl $(1, 1)$.

Lemma 2.2.3 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. If H has Joint Chirality-Curl $(1, 1)$, then $\theta_i \in (0, \pi)$ for all $i \in 1, 2, 3$ and $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$.*

Proof: Let $H \in Equ_0(6)$ be parametrized with action-angle coordinates $(d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ arising from the T_{135} triangulation. If H is a right-handed trefoil with $curl(H) = 1$, then from Lemma 2.1.4 we know $\theta_i \in (0, \pi)$ for all $i \in 1, 2, 3$. If $\theta_i \in (0, \frac{\pi}{2})$ for all $i \in \{1, 2, 3\}$, then clearly $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$. Additionally if $\theta_i, \theta_j \in (0, \frac{\pi}{2})$, for any two distinct $i, j \in \{1, 2, 3\}$, then $\theta_i + \theta_j < \pi$.

Next we will show that if $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$ and H has Joint Chirality-Curl $(1, 1)$, then $\theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_3 < \pi$. Towards a contradiction, suppose that $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$, and $\theta_1 + \theta_2 = \pi$. Substituting $\theta_1 = \pi - \theta_2$ into equation g_3 from Proposition 2.2.1 and using the facts that $\cos(\pi - \theta_2) = -\cos(\theta_2)$ and $\sin(\pi - \theta_2) = \sin(\theta_2)$, we obtain the following

$$
\begin{aligned}
g_3 &= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\pi-\theta_2)\sin(\theta_2)+\sin(\pi-\theta_2)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}
$$

We make the same substitutions into equation h_3 to obtain

$$
h_3 = \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_1)}{2d_1}.
$$

If $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\geq 0$, then g_3 is negative. Fix d_1,d_2,d_3, and θ_2, and now consider g_3 as a function of θ_1. Since $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\geq 0$, then $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}>0$. This implies that the derivative of g_3 with respect to θ_1,

$$
\sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}(-\sin(\theta_1))\sin(\theta_2)+\cos(\theta_1)\cos(\theta_2)\Big),
$$

is negative for $\theta_1\in(\frac{\pi}{2},\pi)$ and $\theta_2\in(0,\frac{\pi}{2})$. Since g_3 is negative for $\theta_1=\pi-\theta_2$ and g_3 is decreasing, g_3 is negative for all $\theta_1>\pi-\theta_2$. Next suppose $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1<0$, then h_3 is negative. This means that the plane, P_2, containing vertices v_1,v_2, and v_3 does not separate v_4 and v_5 when $\theta_1=\pi-\theta_2$. Therefore as θ_1 increases to π so that $\theta_1>\pi-\theta_2$, P_2 will not separate v_4 and v_5. Thus h_3 is negative for $\theta_1>\pi-\theta_2$. Since both g_3 and h_3 must be positive for H to a right-handed trefoil with $curl(H)=1$, we have reached a contradiction. Therefore if $\theta_1\in(0,\pi)$, $\theta_2,\theta_3\in(0,\frac{\pi}{2})$, and $\theta_1+\theta_2\geq\pi$, H can not have Joint Chirality-Curl $(1,1)$.

Now suppose that $\theta_1\in(\frac{\pi}{2},\pi)$, $\theta_2,\theta_3\in(0,\frac{\pi}{2})$, and $\theta_1+\theta_3=\pi$. Similar to the previous argument, we will substitute $\theta_1=\pi-\theta_3$ into equation g_2 and use the facts that

$\cos(\pi-\theta_3) = -\cos(\theta_3)$ and $\sin(\pi-\theta_3) = \sin(\theta_3)$. This results in the following:

$$\begin{aligned}
g_2 &= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\theta_1)+\sin(\theta_3)\cos(\theta_1)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\pi-\theta_3)+\sin(\theta_3)\cos(\pi-\theta_3)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

Making the same substitutions into h_2 gives

$$h_2 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}+1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 \leq 0$, then g_2 is negative. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 > 0$, then h_2 is negative. Since both equations must be positive to have Joint Chirality-Curl $(1,1)$, we have reached a contradiction. Therefore if H is a right-handed trefoil with $curl(H) = 1$ and $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$, then $\theta_1+\theta_2 < \pi$ and $\theta_1+\theta_3 < \pi$.

Now we will consider the case when $\theta_2 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$. Towards a contradiction, consider the case where $\theta_1+\theta_2 = \pi$. Substituting $\theta_1 = \pi-\theta_2$ into equation g_3 from Proposition 2.2.1, we obtain the following:

$$\begin{aligned}
g_3 &= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\pi-\theta_2)\sin(\theta_2)+\sin(\pi-\theta_2)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}$$

Again, we make the same substitution into equation h_3 to get that

$$h_3 = \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_1)}{2d_1}.$$

Now since $\theta_2 \in (\frac{\pi}{2}, \pi)$ then $\cos(\theta_2) < 0$. If $1 - \frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2} \geq 0$, then g_3 is negative.

If $1 - \frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2} < 0$, then h_3 is negative. By Proposition 2.2.1, both equations must be positive in order for H to have Joint Chirality-Curl $(1,1)$. Thus $\theta_1 + \theta_2 < \pi$.

Now with $\theta_2 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$, consider the case where $\theta_2 + \theta_3 = \pi$. Substituting $\theta_3 = \pi - \theta_2$ into equation g_1 we get the following:

$$
\begin{aligned}
g_1 &= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\theta_3) + \sin(\theta_2)\cos(\theta_3)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\pi-\theta_2) + \sin(\theta_2)\cos(\pi-\theta_2)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}
$$

Making the same substitution in h_1 yields:

$$
h_1 = \sqrt{4-(d_3)^2}\Big(1 - \frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.
$$

Again $\cos(\theta_2) < 0$ since $\theta_2 \in (\frac{\pi}{2}, \pi)$. Therefore if $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \geq 0$, then g_1 will be negative. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 < 0$, then f_1 will be negative. Since both functions must be positive, then we have reached a contradiction. Therefore if $\theta_2 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$, then $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$ in order for H to have Joint Chirality-Curl $(1,1)$.

Now we consider the third case where $\theta_3 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$. First we will show that $\theta_1 + \theta_3 < \pi$ in order for H have Joint Chirality-Curl $(1,1)$. First we substitute

$\theta_1 = \pi - \theta_3$ into equation g_2. This results in the following:

$$\begin{aligned}
g_2 &= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\theta_1)+\sin(\theta_3)\cos(\theta_1)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\pi-\theta_3)+\sin(\theta_3)\cos(\pi-\theta_3)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

Making the same substitutions into h_2 gives

$$h_2 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}+1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

Since $\theta_3 \in (\frac{\pi}{2}, \pi)$, then $\cos(\theta_3) < 0$. So if $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 \geq 0$, then g_2 is negative. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 < 0$, then h_2 is negative. Since both equations must be positive to have a right-handed, positive curl trefoil, we have reached a contradiction.

Next towards a contradiction, consider the case where $\theta_2 + \theta_3 = \pi$. Substituting $\theta_3 = \pi - \theta_2$ into equation g_1 we get the following:

$$\begin{aligned}
g_1 &= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\theta_3)+\sin(\theta_2)\cos(\theta_3)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\pi-\theta_2)+\sin(\theta_2)\cos(\pi-\theta_2)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}$$

When we make the same substitution into h_1, we obtain the following:

$$h_1 = \sqrt{4-(d_3)^2}\Big(1-\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \leq 0$, then g_1 will be negative. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 > 0$, then f_1 will be negative. Since both functions must be positive, then we have reached a contradiction. Therefore if $\theta_3 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$, then $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$ in order for H to have Joint Chirality-Curl $(1, 1)$. $\square$

Thus for $H \in Equ_0(6)$ to be a right-handed trefoil with $curl(H) = 1$, only one of the dihedral angles can be greater than $\pi/2$ with the additional condition that the sum of any two angles must be less than π. This portion of the cube $[0, 2\pi]^3$ is shown in the following Figure 2.5.

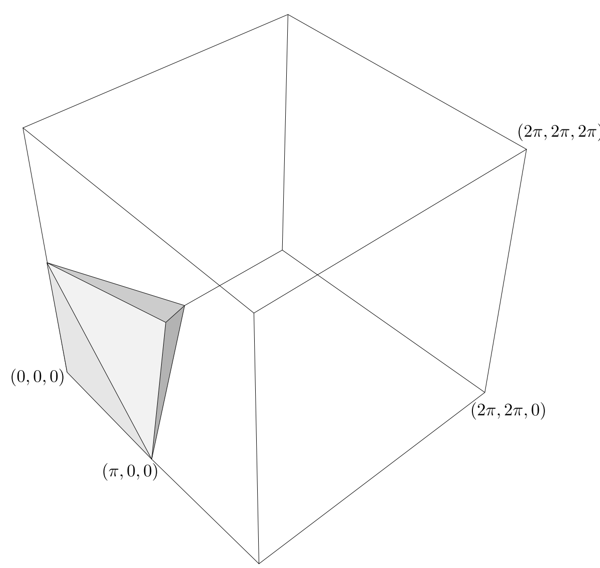


Figure 2.5: The corresponding angles form Lemma 2.2.3 for an equilateral hexagon to have Joint Chirality-Curl $(1, 1)$.

When all three diagonals from the T_{135} triangulation have equal length, H can not have Joint Chirality-Curl $(1, 1)$. So, we continue our analysis with the case where two of the diagonals have equal lengths.

Lemma 2.2.4 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Let $d_1 = d_2$. If H has Joint Chirality-Curl $(1,1)$, then either*

1. $d_3 < d_1 < 1$ *when* $\theta_1 \in (\pi/2, \pi)$, $\theta_2, \theta_3 \in (0, \pi/2)$,

2. $(d_1)^2 < d_3 < d_1\sqrt{4-(d_1)^2}$ *when* $\theta_3 \in (\pi/2, \pi)$, $\theta_1, \theta_2 \in (0, \pi/2)$,

3. $d_3 < (d_1)^2$ *or* $d_1\sqrt{2-d_1} < d_3 < d_1\sqrt{4-(d_1)^2}$ *when* $\theta_i \in (0, \pi/2)$ *for all* i.

Proof: Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Let H be in standard position. Without loss of generality, we assume that $d_1 = d_2$. Consider the case where e_2 and e_3 coincide when $\theta_1 = \theta_2 = 0$. This occurs when $d_3 = d_1\sqrt{4-(d_1)^2}$. If $d_3 \geq d_1\sqrt{4-(d_1)^2}$, then H has Joint Chirality $(0,0)$ for all θ_i. Now we will consider the different ranges of θ_i for H to have Joint Chirality-Curl $(1,1)$ from Lemma 2.2.3.

First suppose that $\theta_1 \in (\pi/2, \pi)$, and $\theta_2, \theta_3 \in (0, \pi/2)$. Let $d_1 > 1$. As θ_3 varies, the orthogonal projection of v_6 on the xy-plane moves along a line passing through v_3 and the midpoint of the segment connecting v_1 and v_5. Since $d_1 > 1$, v_6 has positive x-coordinate and is in the interior of the central triangle with vertices v_1, v_3, and v_5 when $\theta_3 = 0$. Therefore every point on e_6 will have positive x-coordinate for all θ_3. Since $\theta_1 \in (\pi/2, \pi)$, the plane $x = 0$, containing v_1 and v_3, separates e_2 and T_6. Therefore H does not have Joint Chirality-Curl $(1,1)$. Now we will consider the case when $d_1 \leq 1$. First let $d_3 = d_1$. When $\theta_2 = 0$ and $d_3 = d_1$, the altitude of the triangle containing vertices v_3, v_4, and v_5 goes through the origin with e_4 crossing the negative x-axis. Since e_4 must pierce the T_2 for H to have Joint Chirality-Curl $(1,1)$, then $d_3 < d_1$.

Next consider the case when $\theta_2 \in (\pi/2, \pi)$ and $\theta_1, \theta_3 \in (0, \pi/2)$. Since $\theta_2 \in (\pi/2, \pi)$, the plane perpendicular to the xy-plane containing the altitude of the triangle with v_1, v_3, and v_5 separates e_6 and T_4. Therefore e_6 can not pierce T_4 and H can not have Joint Chirality-Curl $(1,1)$.

Now let $\theta_3 \in (\pi/2, \pi)$ and $\theta_1, \theta_2 \in (0, \pi/2)$. First we will consider the case when $d_1 < 1$. Let $\theta_1 = 0$ and consider the isosceles triangle formed by v_1, v_2, and v_3 in the xy-plane. When $d_3 = d_1$, v_5 intersects the altitude of this triangle. Then $d_3 > d_1$ so that e_2 can pierce T_6 for any $\theta_1 \in (0, \pi/2)$ and $\theta_3 \in (\pi/2, \pi)$. Now let $d_1 \geq 1$. Again let $\theta_1 = 0$. Consider the value of d_3 so that v_2 is on the segment connecting v_1 and v_5. Using law of cosines, $d_3 = (d_1)^2$. If e_2 is to pierce T_6, then $d_3 > (d_1)^2$.

Finally, we consider that case where $\theta_i \in (0, \pi/2)$ for all i. We will show that if $(d_1)^2 \leq d_3 \leq d_1\sqrt{2 - d_1}$ then H can not have Joint Chirality-Curl $(1, 1)$. When $\theta_1 = 0$ and $d_3 = (d_1)^2$, e_1 intersects v_5. When $\theta_1 = 0$ and $d_3 = d_1\sqrt{2 - d_1}$, e_2 intersects v_5. An example when $(d_1)^2 \leq d_3 \leq d_1\sqrt{2 - d_1}$ for $\theta_i = 0$ is shown in Figure 2.6.

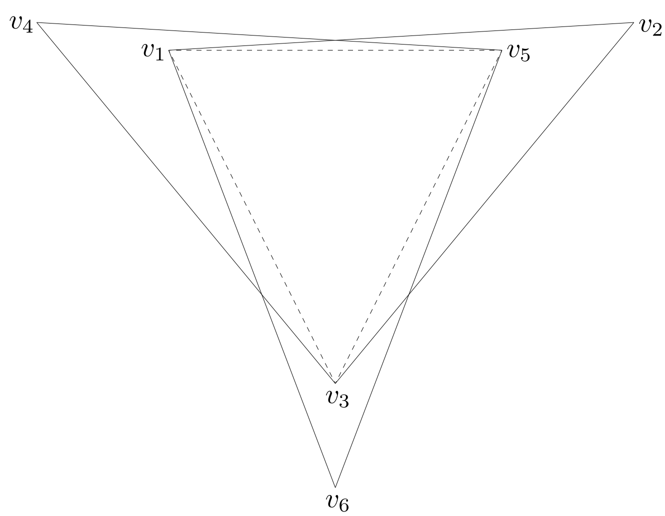


Figure 2.6: With $d_1 = d_2$, $\theta_1 = \theta_2 = 0$, and $(d_1)^2 < d_3 < d_1\sqrt{2 - d_1}$.

First let $\theta_1 = \theta_2$ for $\theta_1, \theta_2 \in (0, \cos^{-1}(\frac{(d_1)^2}{4-(d_1)^2}))$. Since $d_1 = d_2$, when $\theta_1 = \theta_2$ edges e_1 and e_4 intersect. Now consider θ_3. As θ_3 increases from 0 to π there exists one angle, ψ, where e_6 intersects e_3 and e_5 intersects e_2. Since $d_3 \leq d_1\sqrt{2 - d_1}$, then $\psi < \frac{\pi}{2}$. Now keeping θ_2 and θ_3 fixed, increase θ_1 by a small amount δ. Consider the point of intersection of e_2 and e_5 in the orthogonal projection of H onto the xy-plane. Since θ_1 has increased, the z-coordinate of e_2 is greater than the z-coordinate of e_5. Therefore e_2 pierces T_6, as desired for H to have Joint Chirality-Curl $(1,1)$. If θ_1 is decreased then e_2 will not pierce T_6. This means that $\theta_1 > \theta_3$. Similarly θ_2 must be decreased so that e_6 pierces T_4 and $\theta_2 < \theta_3$. Therefore $\theta_1 > \theta_2$ for H to have Joint Chirality-Curl $(1,1)$. Now consider the case when $\theta_1 = \theta_2 = 0$. Since $(d_1)^2 \leq d_3 \leq d_1\sqrt{2-d_1}$ the point of intersection of edges e_1 and e_4 lies exterior of the triangle in the xy-plane with vertices v_1, v_3, and v_5. Let $\theta_1 = \epsilon$ for some small $\epsilon > 0$. When $\theta_2 = 0$, e_4 does not pierce T_2. Since e_4 only intersects e_1 when $\theta_2 = \epsilon$, then e_4 does not pierce T_2 for all $\theta_2 \in (0, \epsilon)$. Therefore $\theta_2 > \theta_1$ for e_4 to pierce T_2 and we have reached a contradiction. Therefore when $d_1 = d_2$ if H has Joint Chirality-Curl $(1,1)$ then either $d_3 < (d_1)^2$ or $d_1\sqrt{2-d_1} < d_3 < d_1\sqrt{4-(d_1)^2}$. □

Next we consider the case where the three diagonals of the T_{135} triangulation are distinct.

Lemma 2.2.5 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Suppose d_1, d_2, and d_3 are distinct and let $d_i > d_j, d_k$. If $J(H) = (1,1)$ then $\theta_i \in (0,\pi)$ and $\theta_j, \theta_k \in (0, \pi/2)$. Moreover, if $d_i > \sqrt{(d_j)^2 + (d_k)^2}$ then $\theta_i \in (\pi/2, \pi)$.*

Proof: Let $H \in Equ_0(6)$ be in standard position so that v_1, v_3, and v_5 are on the xy-plane. Suppose that the lengths of the diagonals are distinct and that $d_2 > d_3 > d_1$. We will show that if H has Joint Chirality-Curl $(1,1)$ then $\theta_2 \in (0,\pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$. Let l_2 be the line in the xy-plane perpendicular to the segment connecting v_1 and v_3 intersecting at the midpoint, m_2. Similarly, we define l_4 and l_6 for segments connecting v_3 to v_5 and v_5 to v_1 respectively. The three lines intersect in a unique point, k, the circumcenter of the triangle with vertices v_1, v_3, and v_5. Moreover, l_i represents the orthogonal projection of v_i onto the xy-plane as θ_i varies and k is the projection of where all vertices coincide, if such point exists.

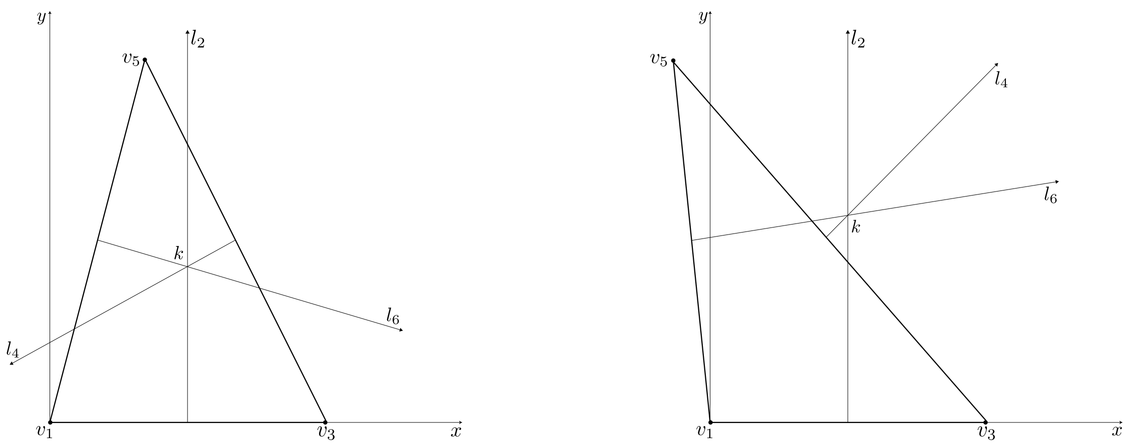


Figure 2.7: The figures show the triangle spanned by vertices (v_1, v_3, v_5) with perpendicular bisector l_i. In the figure on the left, $(d_2)^2 < (d_1)^2 + (d_3)^2$ and so k is interior of the triangle. In the figure on the right, $(d_2)^2 > (d_1)^2 + (d_3)^2$ and so k is exterior of the triangle.

Since $d_3 > d_1$ then l_4 intersects the segment connecting v_1 to v_5 instead of the segment connecting v_1 and v_3. Suppose towards contradiction that $\theta_1 \in (\frac{\pi}{2}, \pi)$. Then the plane perpendicular to the xy-plane containing l_4 separates e_4 and T_2. Therefore H can not have Joint Chirality-Curl $(1, 1)$ if $\theta_1 \in (\frac{\pi}{2}, \pi)$. Let ϕ_1 be the angle for θ_1 where v_2 projects onto k. If $\theta_1 \in (\phi_1, \frac{\pi}{2})$ then e_4 and T_2 are still separated by the plane through l_4. Therefore $\theta_1 \in (0, \phi_1)$. Next suppose that $\theta_3 \in (\frac{\pi}{2}, \pi)$. Since $d_2 > d_3$ the l_2 intersects the segment connecting v_3 and v_5. This means the plane perpendicular to the xy-plane containing l_2 separates e_2 and T_6. Therefore we have reached a contradiction and $\theta_3 \in (0, \frac{\pi}{2})$. Let ϕ_3 be the angle for θ_3 for which v_6 projects onto k. In order for e_2 to pierce T_6 then $\theta_3 \in (0, \phi_3)$. Let ϕ_2 be the angle for θ_2 for which v_4 projects onto k. Let p be the point where e_1 intersects e_4 when $\theta_1 = 0$ and $\theta_2 = \pi$. In order for e_4 to intersect T_2, e_4 must intersect the cone spanned by e_1. The two cones will intersect along an arc connecting p to point which projects onto k. If $\theta_2 < \phi_2$, then e_4 no longer intersects the cone spanned by e_1. Then $\theta_2 \in (\phi_2, \pi)$ for H to have Joint Chirality-Curl $(1,1)$. If $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then the triangle with vertices v_1, v_3, and v_5 is obtuse, as shown in Figure 2.7. Therefore k is exterior of the triangle and $\phi_2 > \frac{\pi}{2}$. Hence if $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$ and $\theta_2 \in (\frac{\pi}{2}, \pi)$.

The moment polytope corresponding to the T_{135} triangulation is split into three equal regions, depending on the which diagonal length is largest. The function $d_1 = \sqrt{(d_2)^2 + (d_3)^2}$ divides each third into two regions, one for acute triangles and one for obtuse triangles, as shown in Figure 2.8.

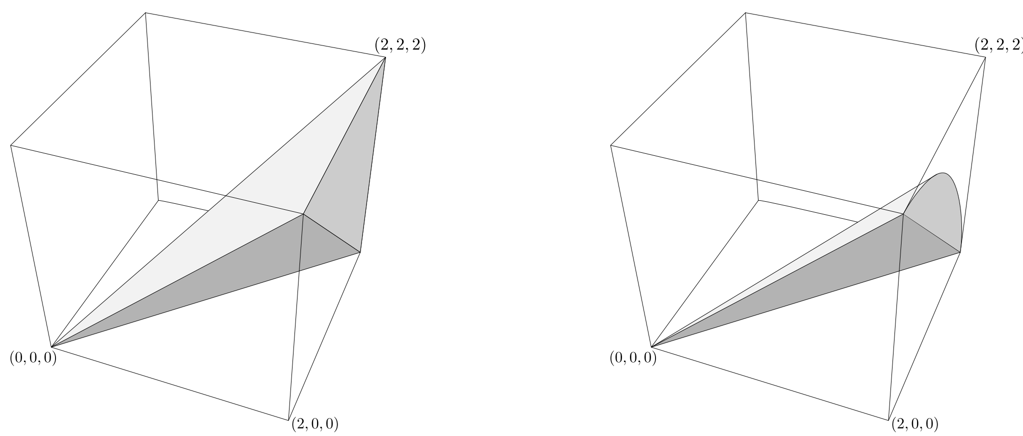


Figure 2.8: The figure on the left shows the portion of the moment polytope, P_6, where $d_1 > d_2, d_3$. The figure on the left shows the portion of the moment polytope where additionally $(d_1)^2 > (d_2)^2 + (d_3)^2$.

2.3 Equilateral, Right-Handed, Negative Curl, Hexagonal Trefoils

In this section, we will prove constraints on action-angle coordinates so that an equilateral hexagon has Joint Chirality-Curl $(1, -1)$. We will start by defining a collection of functions similar to Proposition 2.2.1.

Proposition 2.3.1 *Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates arising from the T_{135} triangulation. If $J(H) = (1, -1)$, then the $f_i > 0$, $g_i < 0$ and $h_i < 0$, for all i, where*

$$
\begin{aligned}
f_1 =& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_3 d - ((d_1)^2 - (d_2)^2 + (d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big) - \\
& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_2 d - ((d_1)^2 + (d_2)^2 - (d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big), \\
g_1 =& \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2 + (d_2)^2 + (d_3)^2}{2d_2d_3}cos(\theta_2)sin(\theta_3) + sin(\theta_2)cos(\theta_3)\Big) - \frac{dsin(\theta_3)}{2d_3}, \\
h_1 =& \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2 + (d_2)^2 + (d_3)^2}{2d_2d_3}cos(\theta_3)sin(\theta_2) + sin(\theta_3)cos(\theta_2)\Big) - \frac{dsin(\theta_2)}{2d_2}, \\
f_2 =& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_1 d - ((d_1)^2 + (d_2)^2 - (d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big) - \\
& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_3 d - (-(d_1)^2 + (d_2)^2 + (d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big), \\
g_2 =& \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2 - (d_2)^2 + (d_3)^2}{2d_1d_3}cos(\theta_3)sin(\theta_1) + sin(\theta_3)cos(\theta_1)\Big) - \frac{dsin(\theta_1)}{2d_1}, \\
h_2 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2 - (d_2)^2 + (d_3)^2}{2d_1d_3}cos(\theta_1)sin(\theta_3) + sin(\theta_1)cos(\theta_3)\Big) - \frac{dsin(\theta_3)}{2d_3}, \\
f_3 =& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_2 d - (-(d_1)^2 + (d_2)^2 + (d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big) - \\
& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_1 d - ((d_1)^2 - (d_2)^2 + (d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big), \\
g_3 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2}cos(\theta_1)sin(\theta_2) + sin(\theta_1)cos(\theta_2)\Big) - \frac{dsin(\theta_2)}{2d_2}, \\
h_3 =& \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2}cos(\theta_2)sin(\theta_1) + sin(\theta_2)cos(\theta_1)\Big) - \frac{dsin(\theta_1)}{2d_1}.
\end{aligned}
$$

Proof: Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the T_{135} triangulation. If $curl(H) = -1$, then $\theta_i \in (\pi, 2\pi)$ for all i by Lemma 2.1.5. Recall that for a hexagon, $H \in Equ_0(6)$ to be a right-handed trefoil then the algebraic intersection numbers Δ_i have to be equal to one for $i = 2, 4, 6$. First we will consider the constraint $\Delta_4 = 1$. This means that the triangular disk T_4 containing v_3, v_4, and v_5 must be pierced by either the edge e_6 or e_1 so that the orientation on the edge agrees with the orientation on T_4, coming from a right-hand rule. If $\theta_2 \in (\pi, 2\pi)$, then e_1 must pierce T_4 for $\Delta_4 = 1$. In order for the line going through v_1 and v_2 to pierce T_4 the following must be negative:

$$(v_2 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) < 0, \tag{2.13}$$

$$(v_2 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) < 0, \tag{2.14}$$

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) < 0. \tag{2.15}$$

Additionally, the plane containing v_3, v_4, and v_5 must separate v_1 and v_2. Therefore the following must be negative:

$$\big((v_2 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) \cdot \big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) < 0.$$

Since $\theta_i \in (\pi, 2\pi)$, then $(v_2 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1)$ is always negative. Letting H be in standard position and evaluating with action-angle coordinates, the remaining three conditions reduce to $h_3 < 0$, $f_3 > 0$, and $g_3 < 0$, respectively.

Next, for $J(H) = (1, -1)$, it is required that $\Delta_2 = 1$. This means that either e_4 or e_5 pierce T_2. If $\theta_1 \in (\pi, 2\pi)$, then e_5 must pierce T_2 for $\Delta_2 = 1$. In order for the line passing through v_5 and v_6 to go through T_2, the following three equations must be negative:

$$(v_6 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) < 0, \tag{2.16}$$

$$(v_6 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) < 0, \tag{2.17}$$

$$(v_6 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) < 0. \tag{2.18}$$

In addition, the plane containing v_1, v_2, and v_3 must separate v_5 and v_6. Therefore the following must be negative:

$$\big((v_6 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) \cdot \big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) < 0.$$

Similar to the previous case, since $\theta_i \in (\pi, 2\pi)$ then $(v_6 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) < 0$. The remaining three conditions are equivalent to $h_2 < 0$, $f_2 > 0$, and $g_2 < 0$, respectively.

The last condition for H to have Joint Chirality-Curl $(1, -1)$ is that e_3 must pierce T_6. For the line containing v_3 and v_4 to go through T_6 then the three inequalities must be satisfied:

$$(v_4 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) < 0, \tag{2.19}$$

$$(v_4 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) < 0, \tag{2.20}$$

$$(v_4 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) < 0. \tag{2.21}$$

The plane containing v_4, v_5, and v_6 must separate v_3 and v_4. Thus the following expression needs to be negative:

$$\big((v_4 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) \cdot \big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) < 0.$$

For $\theta_i \in (\pi, 2\pi)$, inequality 2.21 is always satisfied. The remaining three conditions reduce to $h_1 < 0$, $f_1 > 0$, and $g_1 < 0$. Therefore if H is in standard position, parametrized using action-angle coordinates from the T_{135} triangulation, and has Joint Chirality-Curl $(1, -1)$, then $f_i > 0$, $g_i < 0$, and $h_i < 0$ for all i. □

If $H \in Equ_0(6)$ has Joint Chirality-Curl $(1, -1)$, then all dihedral angles θ_i from the T_{135} triangulation are in the interval $(\pi, 2\pi)$. Now we prove a tighter constraint on the dihedral angles.

Lemma 2.3.2 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. If H has Joint Chirality-Curl $(1, -1)$, then $\theta_i \in (\pi, 2\pi)$ for all $i \in 1, 2, 3$ and $\theta_1 + \theta_2 > 3\pi$, $\theta_1 + \theta_3 > 3\pi$, and $\theta_2 + \theta_3 > 3\pi$.*

Proof: Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the T_{135} triangulation. If the $curl(H) = -1$, then $\theta_i \in (\pi, 2\pi)$ for all i by 2.1.5. If $\theta_i \in (\frac{3\pi}{2}, 2\pi)$ for all i, then $\theta_1 + \theta_2 > 3\pi$, $\theta_1 + \theta_3 > 3\pi$, and $\theta_2 + \theta_3 > 3\pi$. In addition, if $\theta_i, \theta_j \in (\frac{3\pi}{2}, 2\pi)$ for $i, j \in 1, 2$, then $\theta_i + \theta_j > 3\pi$. Therefore we will consider three cases, each where one angle is in the interval $(\pi, \frac{3\pi}{2})$.

First suppose that $\theta_1 \in (\pi, \frac{3\pi}{2})$ and $\theta_2, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$. We will show that if H has Joint Chirality-Curl $(1, -1)$ then $\theta_1 + \theta_2 > 3\pi$ and $\theta_1 + \theta_3 > 3\pi$. Towards a contradiction, suppose that $\theta_1 + \theta_2 = 3\pi$. By substituting $\theta_1 = 3\pi - \theta_2$ into equation g_3 from Proposition

2.3.1, we get

$$g_3 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

Making the same substitution into equation h_3 we get

$$h_3 = \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_1}.$$

Since $\theta_2 \in (\frac{3\pi}{2}, 2\pi)$, then $\cos(\theta_2) > 0$ and $\sin(\theta_2) < 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 \geq 0$, then $g_3 > 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 < 0$, then $h_3 > 0$. Since both h_3 and g_3 must be negative for H to have Joint Chirality-Curl $(1,-1)$, we have reached a contradiction. Now suppose that $\theta_1 \in (\pi, \frac{3\pi}{2})$ and $\theta_2, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$ and $\theta_1+\theta_3 = 3\pi$. Similar to the previous argument, we substitute $\theta_1 = 3\pi - \theta_3$ into equation g_2 to obtain

$$g_2 = \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_1}.$$

Substituting $\theta_1 = 3\pi - \theta_3$ into equation h_2 results in the following

$$h_2 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}+1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1 \leq 0$, then $g_2 > 0$. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1 > 0$, then $h_2 > 0$. Both functions must be negative for H to have Joint Chirality-Curl $(1,-1)$.

Next we will consider the case when $\theta_2 \in (\pi, \frac{3\pi}{2})$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$. Suppose that $\theta_1+\theta_2 = 3\pi$. By making the substitution $\theta_1 = 3\pi - \theta_2$ into g_3 and h_3 we get

$$g_3 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}$$

and

$$h_3 = \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_1}.$$

Since $\theta_2 \in (\pi, \frac{2\pi}{2})$, then $\cos(\theta_2) < 0$. Therefore if $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 \geq 0$ then $g_3 > 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 < 0$ then $h_3 > 0$. Since both g_3 and h_3 must be negative for H to have Joint Chirality-Curl $(1,-1)$, then $\theta_1 + \theta_2 > 3\pi$.

Now with $\theta_2 \in (\pi, \frac{3\pi}{2})$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$, towards a contradiction, assume that $\theta_2 + \theta_3 = 3\pi$. By substituting $\theta_3 = 3\pi - \theta_2$ into g_1 and h_1 we get

$$g_1 = \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_3}$$

and

$$h_1 = \sqrt{4-(d_3)^2}\Big(1 - \frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \geq 0$, then $g_1 > 0$. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 < 0$, then $h_1 < 0$. Since both g_1 and h_1 must be negative, we have reached a contradiction.

Next we consider the case where $\theta_3 \in (\pi, \frac{2\pi}{2})$ and $\theta_1, \theta_2 \in (\frac{3\pi}{2})$. First we will show that $\theta_1 + \theta_3 > 3\pi$. Suppose that $\theta_1 + \theta_3 = 3\pi$. If we substitute $\theta_1 = 3\pi - \theta_2$ into g_2 and h_2 we get

$$g_2 = \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_1}$$

and

$$h_2 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} + 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 \geq 0$, then $g_2 > 0$. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 < 0$, then $h_2 > 0$. Since both g_2 and h_2 must be negative, $\theta_1 + \theta_3 > 2\pi$. Now suppose that $\theta_2 + \theta_3 = 3\pi$. By substituting $\theta_3 = 3\pi - \theta_2$ into g_1 and h_1 we get

$$g_1 = \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_3}$$

and

$$h_1 = \sqrt{4-(d_3)^2}\Big(1 - \frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

Since $\theta_2 \in (\frac{3\pi}{2}, 2\pi)$, then $\cos(\theta_2) > 0$. Therefore if $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \leq 0$, then $g_1 > 0$. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 > 0$, then $h_1 > 0$. Both g_1 and h_1 must be negative for H to have Joint Chirality-Curl $(1, -1)$. Therefore $\theta_2 + \theta_3 > 3\pi$. □

Lemma 2.3.3 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Suppose $d_1, d_2,$ and d_3 are distinct and let $d_i > d_j, d_k$. If $J(H) = (1, -1)$ then $\theta_i \in (\pi, 2\pi)$ and $\theta_j, \theta_k \in (3\pi/2, 2\pi)$. Moreover if $d_i > \sqrt{(d_j)^2 + (d_k)^2}$, then $\theta_i \in (\pi, 3\pi/2)$.*

Proof: Let $H \in Equ_0(6)$ be in standard position so that v_1, v_3, and v_5 are on the xy-plane. Suppose that the lengths of the diagonals are distinct and that $d_2 > d_1 > d_3$. We will show that if H has Joint Chirality-Curl $(1, -1)$ then $\theta_2 \in (\pi, 2\pi)$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$. Let l_2 be the perpendicular bisector to the segment connecting v_1 and v_3. Similarly, we define l_4 and l_6 to be the perpendicular bisectors to segments connecting v_3 to v_5 and v_5 to v_1, respectively. The three lines intersect in a unique point, k, the circumcenter of the triangle spanned by (v_1, v_3, v_5). The orthogonal projection of v_i onto the xy-plane lies on l_i. In addition, k is the orthogonal projection of where all vertices coincide, if such point exists.

Since $d_1 > d_3$, then l_4 intersects the segment connecting v_1 to v_3 instead of the segment connecting v_1 and v_5. Suppose towards contradiction that $\theta_3 \in (\pi, \frac{2\pi}{2})$. Then the plane perpendicular to the xy-plane containing l_4 separates e_3 and T_6. Therefore H can not have Joint Chirality-Curl $(1, -1)$ if $\theta_3 \in (\pi, \frac{3\pi}{2})$.

Next suppose that $\theta_1 \in (\pi, \frac{3\pi}{2})$. Since $d_2 > d_1$ the l_6 intersects the segment connecting v_3 and v_5. This means the plane perpendicular to the xy-plane containing l_6 separates e_5 and T_2. Then H can not have Joint Chirality-Curl $(1, -1)$. Therefore $\theta_1 \in (\frac{3\pi}{2}, 2\pi)$.

Let ψ_3 be the angle for θ_3 for which v_6 projects onto k. If $\theta_3 \in (\pi, \psi_3)$, then the plane perpendicular to the xy-plane containing l_4 still separates e_3 and T_6. Thus if H has Joint Chirality-Curl $(1, -1)$, then $\theta_3 \in (\psi_3, 2\pi)$. Let ψ_1 be the angle of θ_1 so that v_1 projects onto k. If e_5 is to intersect T_2, then $\theta_1 \in (\psi_1, 2\pi)$. Let ψ_2 be the angle for θ_2 for which v_4 projects onto k. Since $\theta_1 \in (\psi_1, 2\pi)$ and $\theta_3 \in (\psi_3, 2\pi)$ then $\theta_2 \in (\psi_2, 2\pi)$ for H to have Joint Chirality-Curl $(1, -1)$. If $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then the triangle spanned by (v_1, v_3, v_5) is obtuse. Therefore k is exterior of the triangle spanned by (v_1, v_3, v_5) and $\psi_2 < \frac{3\pi}{2}$. Hence if $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$ and $\theta_2 \in (\pi, \frac{3\pi}{2})$. $\square$

2.4 Equilateral, Left-Handed, Positive Curl, Hexagonal Trefoils

In this section, we will discuss constraints for H to be a left-handed trefoil with positive curl.

Proposition 2.4.1 *Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates arising from the T_{135} triangulation. If $J(H) = (-1, 1)$, then $f_i < 0$, $g_i > 0$, and $h_i > 0$, for all i, where*

$$
\begin{aligned}
f_1 =& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_3d-((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big)- \\
& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_2d-((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big), \\
g_1 =& \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_2)sin(\theta_3)+sin(\theta_2)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3}, \\
h_1 =& \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_3)sin(\theta_2)+sin(\theta_3)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2}, \\
f_2 =& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_1d-((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big)- \\
& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_3d-(-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big), \\
g_2 =& \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_3)sin(\theta_1)+sin(\theta_3)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1}, \\
h_2 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_1)sin(\theta_3)+sin(\theta_1)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3}, \\
f_3 =& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_2d-(-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big)- \\
& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_1d-((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big), \\
g_3 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_1)sin(\theta_2)+sin(\theta_1)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2}, \\
h_3 =& \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_2)sin(\theta_1)+sin(\theta_2)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1}.
\end{aligned}
$$

Proof: Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates for the T_{135} triangulation. If $curl(H) = 1$, then $\theta_i \in (0, \pi)$ for all i. Additionally, if H has Joint Chirality-Curl $(-1, 1)$ the all algebraic intersection numbers Δ_i must be negative. First we will consider that condition that $\Delta_4 = -1$, meaning the algebraic intersection of T_4 with H is -1. If $\theta_2 \in (0, \pi)$ this means that e_1 must pierce T_4. Therefore the line going through v_1 and v_2 must pass through T_4 and the following three inequalities must be satisfied:

$$(v_2 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) > 0, \tag{2.22}$$

$$(v_2 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) > 0, \tag{2.23}$$

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) > 0. \tag{2.24}$$

Since $\theta_i \in (0, \pi)$, then $(v_2 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1)$ is always positive. Suppose H is in standard position. Evaluating the remaining two expressions with action-angle coordinates from the T_{135} triangulation results in $h_3 > 0$ and $f_3 < 0$.

Additionally, the plane containing v_3, v_4, and v_5 must separate v_1 and v_2. Therefore the following must be negative:

$$\big((v_2 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) \cdot \big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) < 0.$$

This constraint is equivalent to $g_3 > 0$.

Next it is necessary for $\Delta_2 = -1$. If $\theta_1 \in (0, \pi)$, then e_5 must pierce T_2. For the line through v_5 and v_6 to go through T_2 the three inequalities must be satisfied:

$$(v_6 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) > 0, \tag{2.25}$$

$$(v_6 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) > 0, \tag{2.26}$$

$$(v_6 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0. \tag{2.27}$$

Since $\theta_i \in (0, \pi)$, then $(v_6 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0$ for all possible d_i. Using action-angle coordinates, $h_2 > 0$ and $f_2 < 0$ for $J(H) = (-1, 1)$. In addition, the plane containing v_1, v_2, and v_3 must separate v_5 and v_6. Therefore the following must be negative:

$$\big((v_6 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) \cdot \big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) < 0.$$

Satisfying this inequality is equivalent to $g_2 > 0$.

The final condition for H to have Joint Chirality-Curl $(-1, 1)$ is that $\Delta_6 = -1$. If $\theta_3 \in (0, \pi)$, then e_3 must pierce T_6. For the line containing v_3 and v_4 to go through T_6 then the three inequalities must be satisfied:

$$(v_4 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) > 0, \tag{2.28}$$

$$(v_4 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) > 0, \tag{2.29}$$

$$(v_4 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0. \tag{2.30}$$

If $\theta_i \in (0, \pi)$ then $(v_4 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0$ for al d_i. The other two inequalities being positive is equivalent to $h_1 > 0$ and $f_1 < 0$.

Additionally for $\Delta_6 = -1$, the plane containing v_4, v_5, and v_6 must separate v_3 and v_4. Thus the following expression needs to be negative:

$$\big((v_4 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) \cdot \big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) < 0.$$

This means $g_1 > 0$. Therefore if $H \in Equ_0(6)$ is in standard position and has Joint Chirality-Curl $(-1, 1)$, then $f_i < 0$, $g_i > 0$, and $h_i > 0$ for all i. $\square$

Next we define possible dihedral angles for an equilateral hexagon to have Joint

Chirality-Curl $(-1, 1)$.

Lemma 2.4.2 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. If H has Joint Chirality-Curl $(-1, 1)$, then $\theta_i \in (0, \pi)$ for all $i \in 1, 2, 3$ and $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$.*

Proof: Let $H \in Equ_0(6)$ be parametrized with action-angle coordinates $(d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ arising from the T_{135} triangulation. If H has Joint Chirality-Curl $(-1, 1)$, then from Lemma 2.1.4 we know $\theta_i \in (0, \pi)$ for all $i \in 1, 2, 3$. If $\theta_i \in (0, \frac{\pi}{2})$ for all $i \in \{1, 2, 3\}$, then clearly $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$. Additionally if $\theta_i, \theta_j \in (0, \frac{\pi}{2})$, for any two distinct $i, j \in \{1, 2, 3\}$, then $\theta_i + \theta_j < \pi$.

Next we will show that if $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$ and H has Joint Chirality-Curl $(-1, 1)$, then $\theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_3 < \pi$. Towards a contradiction, suppose that $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$, and $\theta_1 + \theta_2 = \pi$. Substituting $\theta_1 = \pi - \theta_2$ into equation g_3 from Proposition 2.4.1 and using the facts that $\cos(\pi - \theta_2) = -\cos(\theta_2)$ and $\sin(\pi - \theta_2) = \sin(\theta_2)$, we obtain the following

$$\begin{aligned}
g_3 &= \sqrt{4 - (d_1)^2}\Big(\frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2}\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4 - (d_1)^2}\Big(\frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2}\cos(\pi - \theta_2)\sin(\theta_2) + \sin(\pi - \theta_2)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4 - (d_1)^2}\Big(- \frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2} + 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}$$

We make the same substitutions into equation h_3 to obtain

$$\begin{aligned}
h_3 &= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_2)\sin(\theta_1)+\sin(\theta_2)\cos(\theta_1)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_2)\sin(\pi-\theta_2)+\sin(\theta_2)\cos(\pi-\theta_2)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

If $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\geq 0$, then g_3 is negative. Fix d_1, d_2, d_3, and θ_2, and now consider g_3 as a function of θ_1. Since $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\geq 0$, then $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}>0$. This implies that the derivative of g_3 with respect to θ_1,

$$\sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}(-\sin(\theta_1))\sin(\theta_2)+\cos(\theta_1)\cos(\theta_2)\Big),$$

is negative for $\theta_1\in(\frac{\pi}{2},\pi)$ and $\theta_2\in(0,\frac{\pi}{2})$. Since g_3 is negative for $\theta_1=\pi-\theta_2$ and g_3 is decreasing, g_3 is negative for all $\theta_1>\pi-\theta_2$. Next suppose $\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1<0$, then h_3 is negative. Thus h_3 is negative for $\theta_1>\pi-\theta_2$. Since both g_3 and h_3 must be positive for H to have Joint Chirality-Curl $(-1,1)$, we have reached a contradiction. Therefore if $\theta_1\in(0,\pi)$, $\theta_2,\theta_3\in(0,\frac{\pi}{2})$, and $\theta_1+\theta_2\geq\pi$, H can not have Joint Chirality-Curl $(-1,1)$.

Now suppose that $\theta_1\in(\frac{\pi}{2},\pi)$, $\theta_2,\theta_3\in(0,\frac{\pi}{2})$, and $\theta_1+\theta_3=\pi$. Similar to the previous argument, we will substitute $\theta_1=\pi-\theta_3$ into equation g_2 and use the facts that $\cos(\pi-\theta_3)=-\cos(\theta_3)$ and $\sin(\pi-\theta_3)=\sin(\theta_3)$. This results in the following:

$$\begin{aligned}
g_2 &= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\theta_1)+\sin(\theta_3)\cos(\theta_1)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\pi-\theta_3)+\sin(\theta_3)\cos(\pi-\theta_3)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1\Big)\cos(\theta_3)\sin(\theta_3)-\frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

Making the same substitutions into h_2 gives

$$\begin{aligned}
h_2 &= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_1)\sin(\theta_3)+\sin(\theta_1)\cos(\theta_3)\Big)-\frac{d\sin(\theta_3)}{2d_3}\\
&= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\pi-\theta_3)\sin(\theta_3)+\sin(\pi-\theta_3)\cos(\theta_3)\Big)-\frac{d\sin(\theta_3)}{2d_3}\\
&= \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}+1\Big)\cos(\theta_3)\sin(\theta_3)-\frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1\leq 0$, then g_2 is negative. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1>0$, then h_2 is negative. Since both equations must be positive to have Joint Chirality-Curl $(-1,1)$, we have reached a contradiction. Therefore if H is a left-handed trefoil with $curl(H)=1$ and $\theta_1\in(\frac{\pi}{2},\pi)$, $\theta_2,\theta_3\in(0,\frac{\pi}{2})$, then $\theta_1+\theta_2<\pi$ and $\theta_1+\theta_3<\pi$.

Now we will consider the case when $\theta_2\in(\frac{\pi}{2},\pi)$ and $\theta_1,\theta_3\in(0,\frac{\pi}{2})$. Towards a contradiction, consider the case where $\theta_1+\theta_2=\pi$. Substituting $\theta_1=\pi-\theta_2$ into equation g_3 from Proposition 2.4.1, we obtain the following:

$$\begin{aligned}
g_3 &= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\pi-\theta_2)\sin(\theta_2)+\sin(\pi-\theta_2)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}$$

We make the same substitutions into equation h_3 to obtain

$$\begin{aligned}
h_3 &= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_2)\sin(\theta_1)+\sin(\theta_2)\cos(\theta_1)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\cos(\theta_2)\sin(\pi-\theta_2)+\sin(\theta_2)\cos(\pi-\theta_2)\Big)-\frac{d\sin(\theta_1)}{2d_1}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

Now since $\theta_2 \in (\frac{\pi}{2},\pi)$ then $\cos(\theta_2)<0$. If $1-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}\geq 0$, then g_3 is negative. If $1-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}<0$, then h_3 is negative. By Proposition 2.4.1, both equations must be positive in order for H to have Joint Chirality-Curl $(-1,1)$. Thus $\theta_1+\theta_2<\pi$.

Now with $\theta_2 \in (\frac{\pi}{2},\pi)$ and $\theta_1,\theta_3 \in (0,\frac{\pi}{2})$, consider the case where $\theta_2+\theta_3=\pi$. Substituting $\theta_3=\pi-\theta_2$ into equation g_1 we get the following:

$$\begin{aligned}
g_1 &= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\theta_3)+\sin(\theta_2)\cos(\theta_3)\Big)-\frac{d\sin(\theta_3)}{2d_3}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\pi-\theta_2)+\sin(\theta_2)\cos(\pi-\theta_2)\Big)-\frac{d\sin(\theta_3)}{2d_3}\\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}$$

Making the same substitution in h_1 yields:

$$\begin{aligned}
h_1 &= \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_3)\sin(\theta_2)+\sin(\theta_3)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\pi-\theta_2)\sin(\theta_2)+\sin(\pi-\theta_2)\cos(\theta_2)\Big)-\frac{d\sin(\theta_2)}{2d_2}\\
&= \sqrt{4-(d_3)^2}\Big(1-\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2)-\frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}$$

Again $\cos(\theta_2)<0$ since $\theta_2 \in (\frac{\pi}{2},\pi)$. Therefore if $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1\geq 0$, then g_1 will be negative. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 < 0$, then f_1 will be negative. Since both functions must be positive, then we have reached a contradiction. Therefore if $\theta_2 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$, then $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$ in order for H to have Joint Chirality-Curl $(-1, 1)$.

Now we consider the third case where $\theta_3 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$. First we will show that $\theta_1 + \theta_3 < \pi$ in order for H have Joint Chirality-Curl $(-1, 1)$. First we substitute $\theta_1 = \pi - \theta_3$ into equation g_2. This results in the following:

$$\begin{aligned}
g_2 &= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\theta_1) + \sin(\theta_3)\cos(\theta_1)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_3)\sin(\pi-\theta_3) + \sin(\theta_3)\cos(\pi-\theta_3)\Big) - \frac{d\sin(\theta_1)}{2d_1} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_1)}{2d_1}.
\end{aligned}$$

Making the same substitutions into h_2 gives

$$\begin{aligned}
h_2 &= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\theta_1)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_3)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}\cos(\pi-\theta_3)\sin(\theta_3) + \sin(\pi-\theta_3)\cos(\theta_3)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} + 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}$$

Since $\theta_3 \in (\frac{\pi}{2}, \pi)$, then $\cos(\theta_3) < 0$. So if $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 \geq 0$, then g_2 is negative. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 < 0$, then h_2 is negative. Since both equations must be positive to have a right-handed, positive curl trefoil, we have reached a contradiction.

Next towards a contradiction, consider the case where $\theta_2 + \theta_3 = \pi$. Substituting $\theta_3 = \pi - \theta_2$ into equation g_1 we get the following:

$$\begin{aligned}
g_1 &= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\theta_3)+\sin(\theta_2)\cos(\theta_3)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_2)\sin(\pi-\theta_2)+\sin(\theta_2)\cos(\pi-\theta_2)\Big) - \frac{d\sin(\theta_3)}{2d_3} \\
&= \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_3)}{2d_3}.
\end{aligned}$$

Making the same substitution in h_1 yields:

$$\begin{aligned}
h_1 &= \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\theta_3)\sin(\theta_2)+\sin(\theta_3)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}\cos(\pi-\theta_2)\sin(\theta_2)+\sin(\pi-\theta_2)\cos(\theta_2)\Big) - \frac{d\sin(\theta_2)}{2d_2} \\
&= \sqrt{4-(d_3)^2}\Big(1 - \frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.
\end{aligned}$$

If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \leq 0$, then g_1 will be negative. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 > 0$, then h_1 will be negative. Since both functions must be positive, then we have reached a contradiction. Therefore if $\theta_3 \in (\frac{\pi}{2}, \pi)$ and $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$, then $\theta_1 + \theta_2 < \pi$, $\theta_1 + \theta_3 < \pi$, and $\theta_2 + \theta_3 < \pi$ in order for H to have Joint Chirality-Curl $(-1, 1)$. $\square$

Next we consider the case where the three diagonals of the T_{135} triangulation are distinct.

Lemma 2.4.3 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Suppose $d_1, d_2,$ and d_3 are distinct and let $d_i > d_j, d_k$. If $J(H) = (-1, 1)$ then $\theta_i \in (0, \pi)$ and $\theta_j, \theta_k \in (0, \pi/2)$. Moreover, if $d_i > \sqrt{(d_j)^2 + (d_k)^2}$ then $\theta_i \in (\pi/2, \pi)$.*

Proof: Let $H \in Equ_0(6)$ be in standard position so that v_1, v_3, and v_5 are on the xy-plane. Suppose that the lengths of the diagonals are distinct and that $d_2 > d_1 > d_3$. We will show that if H has Joint Chirality-Curl $(-1, 1)$ then $\theta_2 \in (0, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$. Let l_2 be perpendicular bisector to the segment connecting v_1 and v_3. Similarly, we define l_4 and l_6 to be the perpendicular bisectors to segments connecting v_3 to v_5 and v_5 to v_1, respectively. The three lines intersect in a unique point, k, the circumcenter of the triangle spanned by (v_1, v_3, v_5). The orthogonal projection of v_i onto the xy-plane lies on l_i. In addition, k is the orthogonal projection of where all vertices coincide, if such point exists.

Since $d_1 > d_3$ then l_4 intersects the segment connecting v_1 to v_3 instead of the segment connecting v_1 and v_5. Suppose towards contradiction that $\theta_3 \in (\frac{\pi}{2}, \pi)$. Then the plane perpendicular to the xy-plane containing l_4 separates e_3 and T_6. Therefore H can not have Joint Chirality-Curl $(-1, 1)$ if $\theta_3 \in (\frac{\pi}{2}, \pi)$.

Next suppose that $\theta_1 \in (\frac{\pi}{2}, \pi)$. Since $d_2 > d_1$ the l_6 intersects the segment connecting v_3 and v_5. This means the plane perpendicular to the xy-plane containing l_6 separates e_5 and T_2. Therefore we have reached a contradiction and $\theta_1 \in (0, \frac{\pi}{2})$.

Let ψ_3 be the angle for θ_3 for which v_6 projects onto k. If $\theta_3 \in (\psi_3, \pi)$, then the plane perpendicular to the xy-plane containing l_4 still separates e_3 and T_6. Thus if H has Joint Chirality-Curl $(-1, 1)$, then $\theta_3 \in (0, \psi_3)$. Let ψ_1 be the angle of θ_1 so that v_1 projects onto k. If e_5 is to intersect T_2, then $\theta_1 \in (0, \psi_1)$. Let ψ_2 be the angle for θ_2 for which v_4 projects onto k. Since $\theta_1 \in (0, \psi_1)$ and $\theta_3 \in (0, \psi_3)$ then $\theta_2 \in (\psi_2, \pi)$ for H to have Joint Chirality-Curl $(-1, 1)$. If $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then the triangle spanned by (v_1, v_3, v_5) is obtuse. Therefore k is exterior of the triangle spanned by (v_1, v_3, v_5) and $\psi_2 > \frac{\pi}{2}$. Hence if $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$ and $\theta_2 \in (\frac{\pi}{2}, \pi)$. $\square$

2.5 Equilateral, Left-Handed, Negative Curl, Hexagonal Trefoils

In this section, we will prove constraints on action-angle coordinates so that an equilateral hexagon has Joint Chirality-Curl $(-1,-1)$.

Proposition 2.5.1 *Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates arising from the T_{135} triangulation. If $J(H) = (-1,-1)$, then the following nine functions must be negative:*

$$\begin{aligned}
f_1 =& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_3d-((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big)-\\
& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_2d-((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big),\\
g_1 =& \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_2)sin(\theta_3)+sin(\theta_2)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3},\\
h_1 =& \sqrt{4-(d_3)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_2d_3}cos(\theta_3)sin(\theta_2)+sin(\theta_3)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2},\\
f_2 =& d_3\sqrt{4-(d_3)^2}sin(\theta_3)\big(d_1d-((d_1)^2+(d_2)^2-(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1))-\\
& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_3d-(-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_3)^2}cos(\theta_3)\big),\\
g_2 =& \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_3)sin(\theta_1)+sin(\theta_3)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1},\\
h_2 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}cos(\theta_1)sin(\theta_3)+sin(\theta_1)cos(\theta_3)\Big)-\frac{dsin(\theta_3)}{2d_3},\\
f_3 =& d_1\sqrt{4-(d_1)^2}sin(\theta_1)\big(d_2d-(-(d_1)^2+(d_2)^2+(d_3)^2)\sqrt{4-(d_2)^2}cos(\theta_2)\big)-\\
& d_2\sqrt{4-(d_2)^2}sin(\theta_2)\big(d_1d-((d_1)^2-(d_2)^2+(d_3)^2)\sqrt{4-(d_1)^2}cos(\theta_1)\big),\\
g_3 =& \sqrt{4-(d_1)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_1)sin(\theta_2)+sin(\theta_1)cos(\theta_2)\Big)-\frac{dsin(\theta_2)}{2d_2},\\
h_3 =& \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}cos(\theta_2)sin(\theta_1)+sin(\theta_2)cos(\theta_1)\Big)-\frac{dsin(\theta_1)}{2d_1}.
\end{aligned}$$

Proof: Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the T_{135} triangulation. If $curl(H) = -1$, then $\theta_i \in (\pi, 2\pi)$ for all i by Lemma 2.1.5. Recall that for a hexagon, $H \in Equ_0(6)$ to be a left-handed trefoil then the algebraic intersection numbers Δ_i have to be equal to negative one for $i = 2, 4, 6$. First we will consider the condition that $\Delta_4 = -1$. If $\theta_2 \in (\pi, 2\pi)$, then e_6 must pierce T_4 for $\Delta_4 = -1$. In order for the line going through v_1 and v_6 to pierce T_4 the following must be negative:

$$(v_6 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) < 0, \tag{2.31}$$

$$(v_6 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) < 0, \tag{2.32}$$

$$(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) < 0. \tag{2.33}$$

In addition, the plane containing v_3, v_4, and v_5 must separate v_1 from v_6. So the following must be negative

$$\big((v_6 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big)\big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) < 0.$$

Since $\theta_i \in (\pi, 2\pi)$ for all i, $(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1)$ is always negative. Evaluating these three with action-angle coordinates gives three functions f_1, g_1, h_1 that must be negative for H to have Joint Chirality-Curl $(-1, -1)$.

Next, for $J(H) = (-1, -1)$, it is required that $\Delta_2 = -1$. If $\theta_1 \in (\pi, 2\pi)$, then e_4 must pierce T_2 for $\Delta_2 = -1$. In order for the line passing through v_4 and v_5 to go through the interior of the triangular disk T_2, the three equations must be negative:

$$(v_4 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) < 0, \tag{2.34}$$

$$(v_4 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) < 0, \tag{2.35}$$

$$(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) < 0. \tag{2.36}$$

In addition, the plane containing v_1, v_2, and v_3 must separate v_4 from v_5. Thus the following equation must be negative:

$$\big((v_4 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big)\big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) < 0.$$

Similar to the previous case, $(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5)$ is always negative. This three leaves three functions f_2, g_2, and h_2 that must be negative.

Lastly, for a right-handed trefoil $\Delta_6 = -1$. If $\theta_3 \in (\pi, 2\pi)$, then e_2 must piece T_6 for $\Delta_6 = -1$. For the line through v_1 and v_2 to go through the interior of T_6 the following three equations must be positive:

$$(v_2 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) < 0, \tag{2.37}$$

$$(v_2 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) < 0, \tag{2.38}$$

$$(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) < 0. \tag{2.39}$$

In addition, the plane containing v_5, v_6, and v_1 must separate v_2 and v_3. Therefore the following equation must be negative:

$$\big((v_2 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big)\big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) < 0.$$

Since $(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) < 0$ is satisfied, the remaining three functions f_3, g_3, h_3 must be negative for H to have Joint Chirality-Curl $(-1, -1)$. $\square$

Next we define possible dihedral angles for an equilateral hexagon to have Joint Chirality-Curl $(-1, -1)$.

Lemma 2.5.2 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. If H has Joint Chirality-Curl $(-1, -1)$, then $\theta_i \in (\pi, 2\pi)$ for all $i \in 1, 2, 3$ and $\theta_1 + \theta_2 > 3\pi$, $\theta_1 + \theta_3 > 3\pi$, and $\theta_2 + \theta_3 > 3\pi$.*

Proof: Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the T_{135} triangulation. If the $curl(H) = -1$, then $\theta_i \in (\pi, 2\pi)$ for all i by 2.1.5. If $\theta_i \in (\frac{3\pi}{2}, 2\pi)$ for all i, then $\theta_1 + \theta_2 > 3\pi$, $\theta_1 + \theta_3 > 3\pi$, and $\theta_2 + \theta_3 > 3\pi$. In addition, if $\theta_i, \theta_j \in (\frac{3\pi}{2}, 2\pi)$ for $i, j \in 1, 2$, then $\theta_i + \theta_j > 3\pi$. Therefore we will consider three cases, each where one angle is in between π and $\frac{3\pi}{2}$.

First suppose that $\theta_1 \in (\pi, \frac{3\pi}{2})$ and $\theta_2, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$. We will show that if H has Joint Chirality-Curl $(-1, -1)$ then $\theta_1 + \theta_2 > 3\pi$ and $\theta_1 + \theta_3 > 3\pi$. Towards a contradiction, suppose that $\theta_1 + \theta_2 = 3\pi$. By substituting $\theta_1 = 3\pi - \theta_2$ into equation g_3 from Proposition 2.3.1, we get

$$g_3 = \sqrt{4 - (d_1)^2}\Big(- \frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2} + 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

Making the same substitution into equation h_3 we get

$$h_3 = \sqrt{4 - (d_2)^2}\Big(\frac{(d_1)^2 + (d_2)^2 - (d_3)^2}{2d_1d_2} - 1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_1}.$$

Since $\theta_2 \in (\frac{3\pi}{2}, 2\pi)$, then $\cos(\theta_2) > 0$ and $\sin(\theta_2) < 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2} + 1 \geq 0$, then $g_3 > 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2} + 1 < 0$, then $h_3 > 0$. Since both h_3 and g_3 must be negative for H to have Joint Chirality-Curl $(-1, -1)$, we have reached a contradiction. Now suppose that $\theta_1 \in (\pi, \frac{3\pi}{2})$ and $\theta_2, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$ and $\theta_1 + \theta_3 = 3\pi$. Similar to the previous argument, we substitute $\theta_1 = 3\pi - \theta_3$ into equation g_2 to obtain

$$g_2 = \sqrt{4 - (d_3)^2}\Big(\frac{(d_1)^2 - (d_2)^2 + (d_3)^2}{2d_1d_3} - 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_1}.$$

Substituting $\theta_1 = 3\pi - \theta_3$ into equation h_2 results in the following

$$h_2 = \sqrt{4 - (d_1)^2}\Big(- \frac{(d_1)^2 - (d_2)^2 + (d_3)^2}{2d_1d_3} + 1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 \leq 0$, then $g_2 > 0$. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3} - 1 > 0$, then $h_2 > 0$. Both functions must be negative for H to have Joint Chirality-Curl $(-1,-1)$.

Next we will consider the case when $\theta_2 \in (\pi, \frac{3\pi}{2})$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$. Suppose that $\theta_1 + \theta_2 = 3\pi$. By making the substitution $\theta_1 = 3\pi - \theta_2$ into g_3 and h_3 we get

$$g_3 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}$$

and

$$h_3 = \sqrt{4-(d_2)^2}\Big(\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_1}.$$

Since $\theta_2 \in (\pi, \frac{2\pi}{2})$, then $\cos(\theta_2) < 0$. Therefore if $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 \geq 0$ then $g_3 > 0$. If $-\frac{(d_1)^2+(d_2)^2-(d_3)^2}{2d_1d_2}+1 < 0$ then $h_3 > 0$. Since both g_3 and h_3 must be negative for H to have Joint Chirality-Curl $(-1,-1)$, then $theta_1 + \theta_2 > 3\pi$.

Now with $\theta_2 \in (\pi, \frac{3\pi}{2})$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$, towards a contradiction, assume that $\theta_2 + \theta_3 = 3\pi$. By substituting $\theta_3 = 3\pi - \theta_2$ into g_1 and h_1 we get

$$g_1 = \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_3}$$

and

$$h_1 = \sqrt{4-(d_3)^2}\Big(1-\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 \geq 0$, then $g_1 > 0$. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2} - 1 < 0$, then $h_1 < 0$. Since both g_1 and h_1 must be negative, we have reached a contradiction.

Next we consider the case where $\theta_3 \in (\pi, \frac{2\pi}{2})$ and $\theta_1, \theta_2 \in (\frac{3\pi}{2})$. First we will show that $\theta_1 + \theta_3 > 3\pi$. Suppose that $\theta_1 + \theta_3 = 3\pi$. If we substitute $\theta_1 = 3\pi - \theta_2$ into g_2 and h_2 we get

$$g_2 = \sqrt{4-(d_3)^2}\Big(\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_1}$$

and

$$h_2 = \sqrt{4-(d_1)^2}\Big(-\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}+1\Big)\cos(\theta_3)\sin(\theta_3) - \frac{d\sin(\theta_3)}{2d_3}.$$

If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1 \geq 0$, then $g_2 > 0$. If $\frac{(d_1)^2-(d_2)^2+(d_3)^2}{2d_1d_3}-1 < 0$, then $h_2 > 0$. Since both g_2 and h_2 must be negative, $\theta_1+\theta_3 > 2\pi$. Now suppose that $\theta_2+\theta_3 = 3\pi$. By substituting $\theta_3 = 3\pi-\theta_2$ into g_1 and h_1 we get

$$g_1 = \sqrt{4-(d_2)^2}\Big(\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_3}$$

and

$$h_1 = \sqrt{4-(d_3)^2}\Big(1-\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}\Big)\sin(\theta_2)\cos(\theta_2) - \frac{d\sin(\theta_2)}{2d_2}.$$

Since $\theta_2 \in (\frac{3\pi}{2}, 2\pi)$, then $\cos(\theta_2) > 0$. Therefore if $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1 \leq 0$, then $g_1 > 0$. If $\frac{-(d_1)^2+(d_2)^2+(d_3)^2}{2d_1d_2}-1 > 0$, then $h_1 > 0$. Both g_1 and h_1 must be negative for H to have Joint Chirality-Curl $(-1,-1)$. Thus $\theta_2+\theta_3 > 3\pi$.

Therefore if H has Joint Chirality-Curl $(-1,-1)$, then $\theta_i \in (\pi, 2\pi)$ for all $i \in 1,2,3$ and $\theta_1+\theta_2 > 3\pi$, $\theta_1+\theta_3 > 3\pi$, and $\theta_2+\theta_3 > 3\pi$. □

Lemma 2.5.3 *Let $H \in Equ_0(6)$ and parametrize H using action-angle coordinates with the T_{135} triangulation. Suppose d_1, d_2, and d_3 are distinct and let $d_i > d_j, d_k$. If $J(H) = (-1,-1)$ then $\theta_i \in (\pi, 2\pi)$ and $\theta_j, \theta_k \in (3\pi/2, 2\pi)$. Moreover, if $d_i > \sqrt{(d_j)^2+(d_k)^2}$ then $\theta_i \in (\pi, 3\pi/2)$.*

Proof: Let $H \in Equ_0(6)$ be in standard position so that v_1, v_3, and v_5 are on the xy-plane. Suppose that the lengths of the diagonals are distinct and that $d_2 > d_3 > d_1$. We will show that if H has Joint Chirality-Curl $(-1,-1)$ then $\theta_2 \in (\pi, 2\pi)$ and $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$.

Let l_2 be the perpendicular bisector to the segment connecting v_1 and v_3. Similarly, we define l_4 and l_6 to be the perpendicular bisectors to the segments connecting v_3 to v_5 and v_5 to v_1, respectively. The three lines intersect in a unique point, k, the circumcenter of the triangle spanned by (v_1, v_3, v_5). The orthogonal projection of v_i onto the xy-plane lies on l_i. In addition, k is the orthogonal projection of where all three vertices coincide, if such point exists.

Since $d_3 > d_1$, then l_4 intersects the segment connecting v_1 to v_5 instead of the segment connecting v_1 and v_3. Suppose towards contradiction that $\theta_1 \in (\pi, \frac{3\pi}{2})$. Then the plane perpendicular to the xy-plane containing l_4 separates e_4 and T_2. Therefore H can not have Joint Chirality-Curl $(-1,-1)$ if $\theta_1 \in (\pi, \frac{3\pi}{2})$.

Next suppose that $\theta_3 \in (\pi, \frac{3\pi}{2})$. Since $d_2 > d_3$ the l_2 intersects the segment connecting v_3 and v_5. This means the plane perpendicular to the xy-plane containing l_2 separates e_2 and T_6. Therefore e_2 can not pierce T_6 and H can not have Joint Chirality-Curl $(-1,-1)$. Hence $\theta_3 \in (\frac{3\pi}{2}, 2\pi)$.

Let ψ_1 be the angle for θ_1 where v_2 projects onto k. If $\theta_1 \in (\pi, \psi_1)$ then e_4 and T_2 are still separated by the plane through l_4. Therefore $\theta_1 \in (\psi_1, 2\pi)$ for H to have Joint Chirality-Curl $(-1,-1)$. Similarly, let ψ_3 be the angle for θ_3 for which v_6 projects onto k. In order for e_2 to pierce T_6 then $\theta_3 \in (\psi_2, 2\pi)$. Now let ψ_2 be the angle for θ_2 for which v_4 projects onto k. Since $\theta_1 \in (\psi_1, 2\pi)$ and $\theta_3 \in (\psi_3, 2\pi)$ then $\theta_2 \in (\psi_2, 2\pi)$ for H to have Joint Chirality-Curl $(-1,-1)$. If $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then the triangle spanned by (v_1, v_3, v_5) is obtuse. Therefore k is is exterior of the triangle and $\psi_2 < \frac{3\pi}{2}$. Hence if $d_2 > \sqrt{(d_1)^2 + (d_3)^2}$ then $\theta_1, \theta_3 \in (\frac{3\pi}{2}, 2\pi)$ and $\theta_2 \in (\pi, \frac{3\pi}{2})$. $\square$

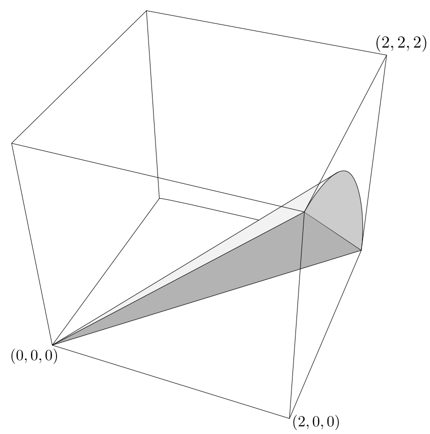
2.6 Knotting Probability of Hexagonal Trefoil

In this section, we will discuss the probability that a random equilateral hexagon is knotted. It has been proven that at least $\frac{1}{3}$ of hexagons with total length 2 are unknotted[2]. Using action-angle coordinates and Calvo's geometric invariant *curl*, Cantarella and Shonkwiler [2] prove that at least $\frac{1}{2}$ of the space of equilateral hexagons consists of unknots. In order to gain intuition on the tightness of these bounds, we perform a Monte Carlo experiment. We randomly sample a point in the moment polytope P_6 and a point in the cube $[0, 2\pi]^3$. We then test whether this point satisfies the necessary constraints to have a hexagonal trefoil with Joint Chirality-Curl $(1, 1)$, described in Proposition 2.2.1. Taking a sample size of 10 million configurations, repeating this experiment multiple times, on average the fraction of $(1, 1)$ trefoils is 3.426005×10^{-5} with standard deviation 2.241511×10^{-6}. Since there are four types of trefoils, we estimate that the knot probability for equilateral hexagons is about 1.370402×10^{-4}. Using the lemmas from the previous sections, we improve the theoretical bound.

Theorem 2.6.1 *The probability that an equilateral hexagon is knotted is at most* $\frac{14-3\pi}{192}$.

Proof: Let $H \in Equ_0(6)$. We will choose the T_{135} triangulation of H to form our set of action-angle coordinates: $\alpha : P_6 \times T^3 \mapsto Pol_0(6)$. Since almost all of $Pol_0(6)$ is a toric symplectic manifold, Theorem 1.2.9 holds for integrals over this space. First we will calculate the expected value for $H = (d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ to have $curl = 1$. Suppose that H is in general position so that the lengths of the diagonals are distinct. Without loss of generality, we assume that d_1 is the largest of the three diagonals. The moment polytope, P_6, corresponding to the T_{135} triangulation is $\frac{1}{2}$ of the cube $[0, 2]^3$. Therefore the volume of P_6 is 4. The region where one diagonal is greater than the other two divides the moment polytope into 3 regions with equal volume of $\frac{4}{3}$. From Lemma 2.2.5 and Lemma 2.4.3 if $d_1 > d_2, d_3$ and $curl(H) = 1$, then $\theta_1 \in (0, \pi)$, $\theta_2 \in (0, \frac{\pi}{2})$, and

$\theta_3 \in (0, \frac{\pi}{2})$. Additionally, if $(d_1)^2 > (d_2)^2 + (d_3)^2$, then $\theta_1 \in (\frac{\pi}{2}, \pi)$. From Lemma 2.2.3 and Lemma 2.4.2, we know that $\theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_3 < \pi$, shown in Figure 2.9



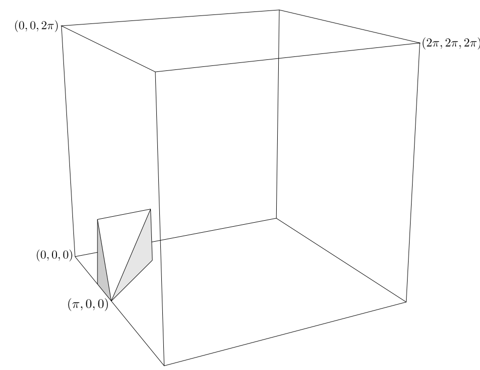


Figure 2.9: The figure on the left shows the portion of the moment polytope P_6 where $d_1 > d_2, d_3$ and $(d_1)^2 > (d_2)^2 + (d_3)^2$. The figure on the right shows the portion of cube $[0, 2\pi]^3$ where $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$, $\theta_1 + \theta_2 < \pi$, and $\theta_1 + \theta_3 < \pi$.

Using standard integration, we calculate that the volume of the portion of P_6 where $d_1 > d_2$, $d_1 > d_3$, and $(d_1)^2 > (d_2)^2 + (d_3)^2$ is equal to $\frac{2(\pi-2)}{3}$. The ratio of this volume out of the third of P_6 is $\frac{\pi}{2} - 1$. The region where $\theta_1 \in (\frac{\pi}{2}, \pi)$, $\theta_2 \in (0, \frac{\pi}{2})$, $\theta_3 \in (0, \frac{\pi}{2})$, $\theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_3 < \pi$ is $\frac{1}{192}$ of the cube $[0, 2\pi]^3$.

The portion of P_6 where $d_1 > d_2, d_3$ and $(d_1)^2 < (d_2)^2 + (d_3)^2$ is $2 - \frac{\pi}{2}$ of the volume of $\frac{1}{3}$ of P_6. The region where $\theta_1 \in (0, \pi)$, $\theta_2 \in (0, \frac{\pi}{2})$, $\theta_3 \in (0, \frac{\pi}{2})$, $\theta_1 + \theta_2 < \pi$ and $\theta_1 + \theta_3 < \pi$, shown in Figure 2.10, is $\frac{1}{48}$ of the cube $[0, 2\pi]^3$.

Therefore the expected value for $curl(H) = 1$ is bounded above by

$$\left(\frac{\pi}{2} - 1\right)\left(\frac{1}{192}\right) + \left(2 - \frac{\pi}{2}\right)\left(\frac{1}{48}\right) = \frac{7}{192} - \frac{\pi}{128}.$$

Making a similar argument for $curl(H) = -1$, we see that the knot probability is at most

$$2(\frac{7}{192} - \frac{\pi}{128}) = \frac{14 - 3\pi}{192},$$

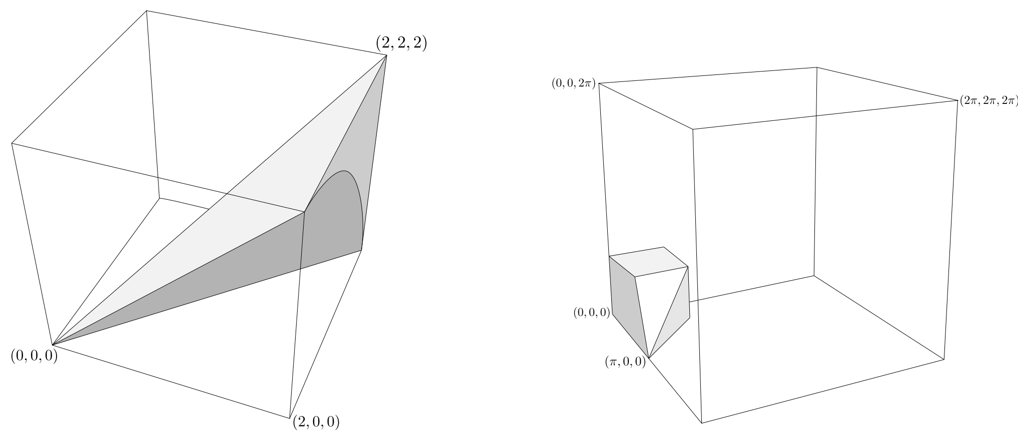


Figure 2.10: The figure on the left shows the portion of the moment polytope P_6 where $d_1 > d_2, d_3$ and $(d_1)^2 < (d_2)^2 + (d_3)^2$. The figure on the right shows the portion of cube $[0, 2\pi]^3$ where $\theta_1 \in (0, \pi)$, $\theta_2, \theta_3 \in (0, \frac{\pi}{2})$, $\theta_1 + \theta_2 < \pi$, and $\theta_1 + \theta_3 < \pi$.

as desired. $\square$

Chapter 3

Symmetric Hexagons

Optimal configurations for certain knot energies or physical properties are often spatially symmetric. For example, Jonathan Simon [14] defines a minimum distance knot energy for polygonal knots with varying edge-length. He conjectures that the minimum hexagonal trefoil knot has a dihedral symmetry, specifically a rotation of period three and orthogonal rotation of period two. In particular, the distance between odd vertices are equal. Therefore these dihedrally symmetric hexagons are unknotted when the polygon is equilateral. Using numerical simulation, we have found that maximally thick equilateral hexagonal trefoil knots appear to have a different spatial symmetry that we will explore in this chapter.

3.1 Symmetric Equilateral Hexagons

We investigate the properties of equilateral hexagons that realize certain symmetries. In particular, we look at symmetries that arise from a reflection in the regular planar hexagon across the line segments connecting midpoints of opposite edges. There are three such reflections: r_1, r_2, and r_3 the reflections along the lines connecting the midpoints of

e_2 and e_5, e_1 and e_4, and e_3 and e_6, respectively.

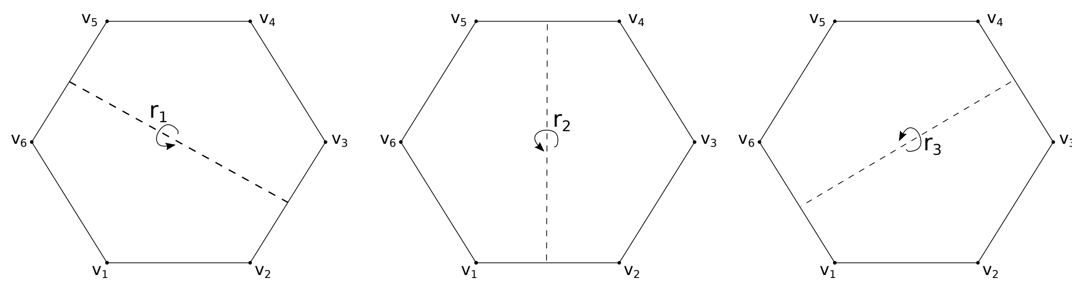


Figure 3.1: Reflections r_1, r_2, and r_3 of an equilateral hexagon.

We will focus on the hexagons whose geometric structure is invariant under the reflection r_1.

Definition 3.1.1 *Let $H \in Equ_0(6)$. If H is invariant under the automorphism that takes vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, in order, to $\{v_4, v_3, v_2, v_1, v_6, v_5\}$, then H is said to be symmetric.*

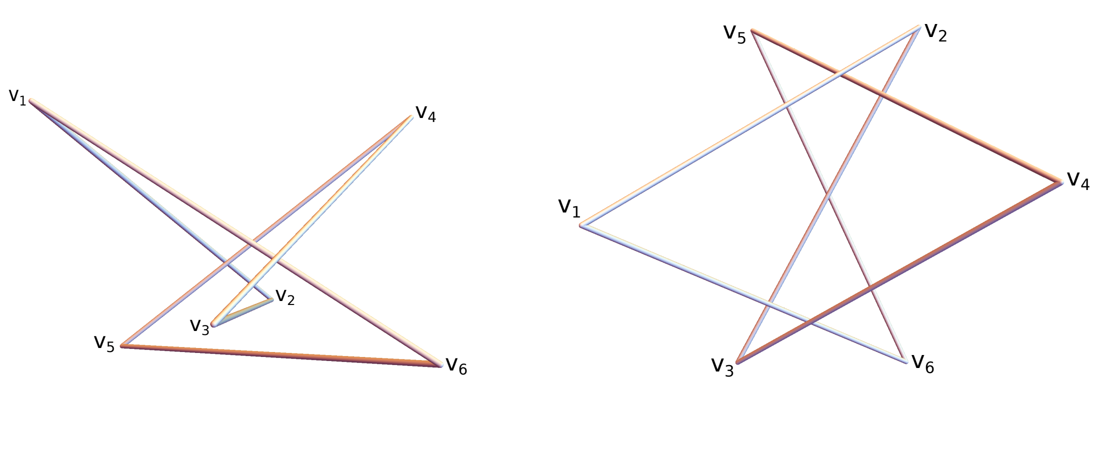


Figure 3.2: Two views of a symmetric equilateral hexagonal trefoil.

If H is symmetric, then the configuration is invariant under a π rotation through the axis intersecting the midpoints of e_2 and e_5. This sends the segments connecting v_1 to v_3, v_3 to v_5, and v_5 to v_1 to the segments connecting v_2 to v_4, v_2 to v_6, and v_4 to v_6, respectively. The corresponding distances, in a 3-space configuration, are equal.

To parametrize the space of equilateral hexagons, it suffices to consider any triangulation of the planar hexagon.

Definition 3.1.2 *The Z triangulation of an equilateral hexagon is the triangulation with diagonals connecting vertices v_2 to v_5, v_3 to v_5, and v_2 to v_6, with lengths z_1, z_2, and z_3, respectively.*

Additionally let ϕ_i be the dihedral angle around diagonal z_i. The Z triangulation is shown in Figure 3.3.

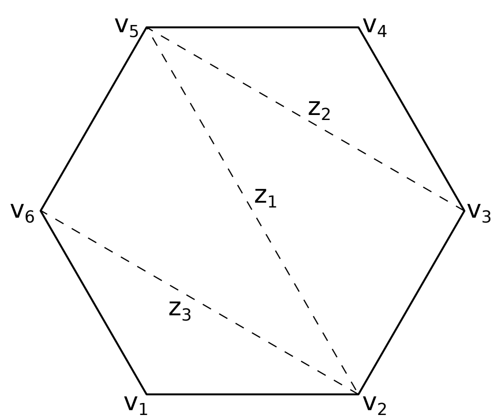


Figure 3.3: This figure shows the Z triangulation of an equilateral hexagon.

The lengths of the diagonals of the Z triangulation must obey the following triangulation inequalities:

$$\begin{aligned} 0 \le\, & z_1 \le 3, & & 1 \le z_1 + z_2, & & 1 \le z_1 + z_3, \\ 0 \le\, & z_2 \le 2, \quad \text{and} & & z_1 \le 1 + z_2, \quad \text{and} & & z_1 \le 1 + z_3, \\ 0 \le\, & z_3 \le 2, & & z_2 \le 1 + z_1, & & z_3 \le 1 + z_1. \end{aligned}$$

Definition 3.1.3 *The Z triangulation polytope, P_z, is the moment polytope for $Pol_0(6)$ corresponding to the Z triangulation and is determined by the triangulation inequalities.*

The Z triangulation polytope is shown in Figure 3.4.

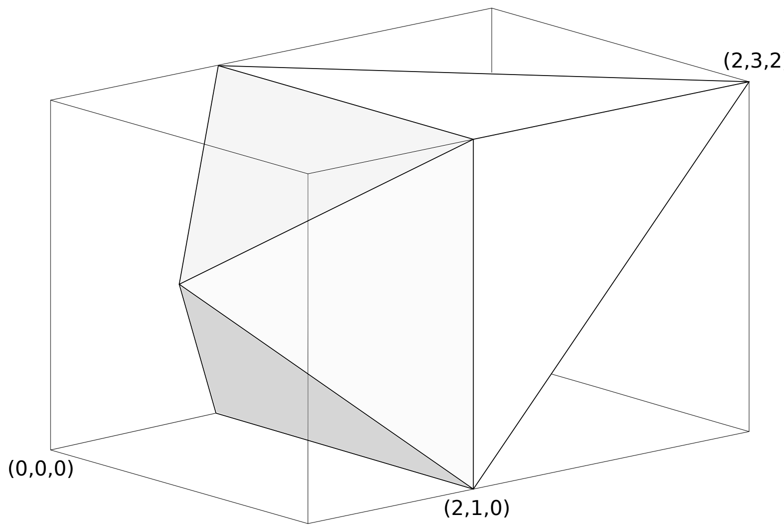


Figure 3.4: The moment polytope corresponding to the Z triangulation of an equilateral hexagon.

For $H \in Equ_0(6)$, we will rotate H so that v_2 is the origin, v_5 is on the positive x-axis, and v_6 is on the xy-plane with v_2, v_5, and v_6 oriented in a counter-clockwise direction. We will consider this the standard position for H when using the Z triangulation. So for parametrizing $H = (z_1, z_2, z_3, \phi_1, \phi_2, \phi_3)$, we have the following coordinates for the vertices of H:

$$
\begin{aligned}
v_1 =&\Big(\frac{-1+(z_1)^2+(z_3)^2}{4z_1} + \frac{\sqrt{4-(z_3)^2}\sqrt{-(z_1)^4-((z_3)^2-1)^2+2(z_1)^2(1+(z_3)^2)}}{4z_1z_3}\cos(\phi_3),\\
&\frac{\sqrt{-(z_1)^4-((z_3)^2-1)^2+2(z_1)^2(1+(z_3)^2)}}{4z_1} - \frac{\sqrt{4-(z_3)^2}(-1+(z_1)^2+(z_3)^2)}{4z_1z_3}\cos(\phi_3),\\
&\frac{1}{2}\sqrt{4-(z_3)^2}\sin(\phi_3)\Big),\\
v_2 =&(0,0,0),\\
v_3 =&\Big(\frac{(z_1)^2+(z_2)^2+1}{2z_1}, \frac{\sqrt{-1+2(z_1)^2-(z_1)^4+2(z_2)^2+2(z_1)^2(z_2)^2-(z_2)^4}}{2z_1}\cos(\phi_1),\\
&\frac{\sqrt{-1+2(z_1)^2-(z_1)^4+2(z_2)^2+2(z_1)^2(z_2)^2-(z_2)^4}}{2z_1}\sin(\phi_1)\Big),\\
v_4 =&\Big(\frac{1+3z_1-z_2}{4z_1} - \frac{\sqrt{4-(z_2)^2}\sqrt{-(z_1)^4-((z_2)^2-1)^2+2(z_1)^2(1+(z_2)^2)}}{4z_1z_2}\cos(\phi_2),\\
&\frac{\sqrt{-(z_1)^4-((z_2)^2-1)^2+2(z_1)^2(1+(z_2)^2)}}{4z_1}\cos(\phi_1)-\\
&\frac{\sqrt{4-(z_2)^2}(-1+(z_1)^2+(z_2)^2)}{4z_1z_2}\cos(\phi_1)\cos(\phi_2) + \frac{1}{2}\sqrt{4-(z_1)^2}\sin(\phi_1)\sin(\phi_2),\\
&\frac{\sqrt{-(z_1)^4-((z_2)^2-1)^2+2(z_1)^2(1+(z_2)^2)}}{4z_1}\sin(\phi_1) + \frac{1}{2}\sqrt{4-(z_2)^2}\cos(\phi_1)\sin(\phi_2)-\\
&\frac{\sqrt{4-(z_1)^2}(-1+(z_1)^2+(z_2)^2)}{4z_1z_2}\cos(\phi_2)\sin(\phi_1)\Big),\\
v_5 =&(z_1,0,0),\\
v_6 =&\Big(\frac{(z_1)^2+(z_3)^2-1}{2z_1}, \frac{\sqrt{-(z_1)^4-((z_3)^2-1)^2+2(z_1)^2(1+(z_3)^2)}}{2z_1}, 0\Big).
\end{aligned}
$$

Recall that the dihedral group of twelve elements that acts on $Equ(6)$ can be generated by automorphism r and s that reverse and shift the order of the vertices. Since $r_1 = s^3 r$, the group, γ, generated by r_1 is a subgroup of $\langle s^2, rs \rangle$. Therefore by Theorem 1.1.15, $Equ(6)/\Gamma$ has five components and there are five types of symmetric hexagons.

Consider $H \in Equ_0(6)$. We derive constraints on action-angle coordinates so that H is symmetric. Recall that H must be invariant under the automorphism that takes vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, in order, to $\{v_4, v_3, v_2, v_1, v_6, v_5\}$. This sends the diagonals from the T_{135} connecting v_1 to v_3, v_3 to v_5, and v_5 to v_1 to the segments connecting v_2 to v_4, v_2 to v_6, and v_4 to v_6. Additionally the segment connecting v_2 to v_5 to the segment connecting v_3 to v_6. Therefore the corresponding distances are equal, resulting in the following set of equations that must be satisfied:

$$||v_3 - v_1|| - ||v_4 - v_2|| = 0, \tag{3.1}$$

$$||v_5 - v_3|| - ||v_6 - v_2|| = 0, \tag{3.2}$$

$$||v_5 - v_1|| - ||v_6 - v_4|| = 0, \tag{3.3}$$

$$||v_5 - v_2|| - ||v_6 - v_3|| = 0. \tag{3.4}$$

Evaluating with action-angle coordinates form the Z triangulation, equation 3.2 reduces to the requirement that $z_2 = z_3$. This gives a new moment polytope for symmetric hexagons shown in Figure 3.5.

Letting H be in standard position and using the relation that $z_2 = z_3$, equation 3.4 results in the following

$$-3(z_1)^4+(-1+(z_2)^2)^2+2(z_1)^2(1+(z_2)^2)+((z_1)^4+(-1+(z_2)^2)^2-2(z_1)^2(1+(z_2)^2))\cos(\phi_1) = 0.$$

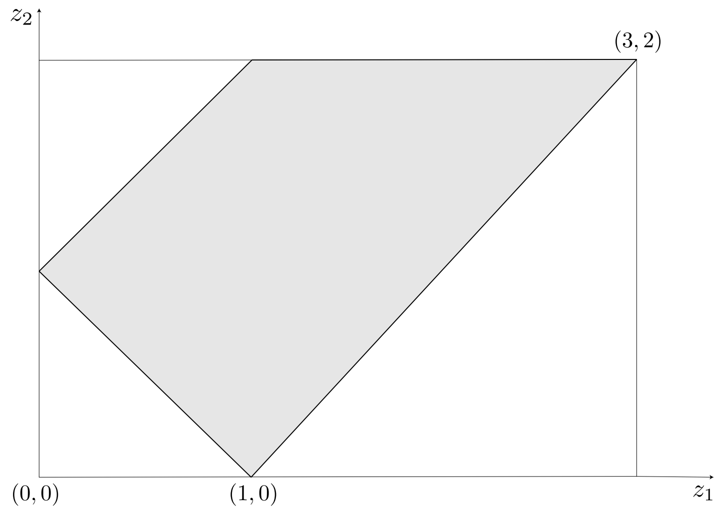


Figure 3.5: This figure shows the moment polytope corresponding to the Z triangulation when $z_2 = z_3$.

Therefore if H is symmetric, then we can define ϕ_1 as a function of z_1 and z_2, which we will denote Φ_1:

$$\Phi_1(z_1, z_2) = \pm\arccos\Big(\frac{3(z_1)^4 - ((z_2)^2 - 1)^2 - 2(z_1)^2(1 + (z_2)^2)}{(z_1)^4 + ((z_2)^2 - 1)^2 - 2(z_1)^2(1 + (z_2)^2)}\Big).$$

We will denote the positive values as $\Phi_1^+(z_1, z_2)$ and the negative values of as $\Phi_1^-(z_1, z_2)$. Continuing in this manner, we can use 3.1 to solve for ϕ_2. Replacing ϕ_1 with $\Phi_1^+(z_1, z_2)$ results in:

$$\Phi_2^+(z_1, z_2, \phi_3) = \pm\arccos\Bigg(\frac{-(1 + (z_1)^4 + 2(z_2)^2 - 3(z_1)^2(-1 + (z_2)^2)\cos(\phi_3)}{(z_1)^4 + (-1 + (z_2)^2)^2 - 2(z_1)^2(1 + (z_2)^2)^2} - \frac{2\sqrt{2}z_2\sqrt{-1 - (z_1)^6 + (z_2)^2 - (z_2)^6 + (z_1)^2(-1 + (z_2)^2)^2 + (z_1)^4(1 + (z_2)^2)}\sin(\phi_3)}{(z_1)^4 + (-1 + (z_2)^2)^2 - 2(z_1)^2(1 + (z_2)^2)^2}\Bigg).$$

We will denote the positive and negative values of ϕ_2 using $\Phi_1^+(z_1, z_2)$ as $\Phi_2^{++}(z_1, z_2, \phi_3)$ and $\Phi_2^{+-}(z_1, z_2, \phi_3)$. Similarly we could use 3.1 to solve for ϕ_2, replacing ϕ_1 with $\Phi_1^-(z_1, z,2)$.

This gives the following:

$$\Phi_2^-(z_1,z_2,\phi_3)=\pm\arccos\left(\frac{-(1+(z_1)^4+2(z_2)^2-3(z_1)^2(-1+(z_2)^2)\cos(\phi_3)}{(z_1)^4+(-1+(z_2)^2)^2-2(z_1)^2(1+(z_2)^2)^2}+\right.$$
$$\left.\frac{2\sqrt{2}z_2\sqrt{-1-(z_1)^6+(z_2)^2-(z_2)^6+(z_1)^2(-1+(z_2)^2)^2+(z_1)^4(1+(z_2)^2)}\sin(\phi_3)}{(z_1)^4+(-1+(z_2)^2)^2-2(z_1)^2(1+(z_2)^2)^2}\right).$$

Again we will denote the positive and negative values for ϕ_2 coming from $\Phi_1^-(z_1,z_2)$ as $\Phi_2^{-+}(z_1,z_2,\phi_3)$ and $\Phi_2^{--}(z_1,z_2,\phi_3)$, respectively.

Lastly, using action-angle coordinates from the Z triangulation and letting H be in standard position, we can evaluate equation 3.3. We can then replace (ϕ_1,ϕ_2) with $(\Phi_1^+(z_1,z_2),\Phi_2^{++}(z_1,z_2,\phi_3))$, $(\Phi_1^+(z_1,z_2),\Phi_2^{+-}(z_1,z_2,\phi_3))$, $(\Phi_1^-(z_1,z_2),\Phi_2^{-+}(z_1,z_2,\phi_3))$, or $(\Phi_1^-(z_1,z_2),\Phi_2^{--}(z_1,z_2,\phi_3))$. This gives four functions, $\varphi_1(z_1,z_2,\phi_3)$, $\varphi_2(z_1,z_2,\phi_3)$, $\varphi_3(z_1,z_2,\phi_3)$, and $\varphi_4(z_1,z_2,\phi_3)$, respectively, that must be equal to zero. Setting each $\varphi_i=0$ and solving for ϕ_3 gives a range of admissible ϕ_3 for H to be symmetric. Therefore we can now parametrize symmetric equilateral hexagons using variables z_1, z_2, and ϕ_3.

We will now describe additional constraints on the lengths of the diagonals for H to be a symmetric equilateral hexagon. These constraints are also necessary for Φ_1^* and Φ_2^{**} to be well defined.

Proposition 3.1.4 *Let $H \in Equ_0(6)$. Let z_1, z_2, and z_3 be the lengths of the diagonals from the Z triangulation. If H is symmetric then $(z_1)^2+(z_2)^2>1$ and $|(z_1)^2-(z_2)^2|<1$.*

Proof: Let $H \in Equ_0(6)$ and let z_i be the lengths of the diagonals from the Z triangulation. If H is symmetric, then $z_2=z_3$. Since the configuration is invariant under a π rotation through the axis intersecting the midpoints of e_2 and e_5, e_2 and e_5 lie on opposite circles of a cylinder of radius $\frac{1}{2}$, as shown in Figure 3.6. Let h be the height of the cylinder.

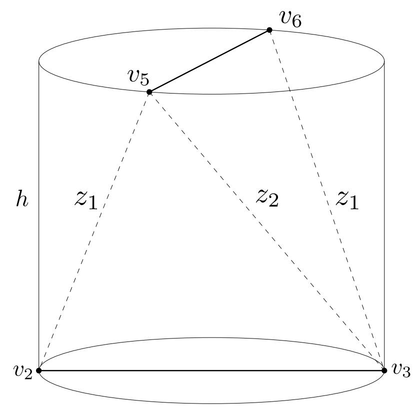


Figure 3.6: If H is symmetric then e_2 and e_5 lie on opposite disks of a cylinder with radius $\frac{1}{2}$.

Orient the cylinder so that the center axis is the z-axis, the center of the bottom disk is the origin and e_2 is on the x-axis. Therefore $v_3 = (1/2, 0, 0)$, $v_2 = (-1/2, 0, 0)$, $v_6 = (a, \sqrt{\frac{1}{4} - a^2}, h)$, and $v_5 = (-a, -\sqrt{\frac{1}{4} - a^2}, h)$, for some a with $-\frac{1}{2} \leq a \leq \frac{1}{2}$. Since the distance from v_3 to v_6 is equal to z_1, then

$$z_1 = \sqrt{\left(a - \frac{1}{2}\right)^2 + \left(\sqrt{\frac{1}{4} - a^2} - 0\right)^2 + \left(h - 0\right)^2} = \sqrt{h^2 + \frac{1}{2} - a}.$$

Since the distance from v_3 to v_5 is equal to z_2, then $z_2 = \sqrt{h^2 + \frac{1}{2} + a}$. Reducing this system by eliminating a we obtain $(z_2)^2 + (z_1)^2 = 2h^2 + 1$. Since $h > 0$, then $(z_1)^2 + (z_2)^2 > 1$, as desired. Reducing this system by eliminating h we obtain that $(z_2)^2 - (z_1)^2 = 2a$. Since $-\frac{1}{2} \leq a \leq \frac{1}{2}$, then $-1 < (z_2)^2 - (z_1)^2 < 1$. Therefore for H to be symmetric then $(z_1)^2 + (z_2)^2 > 1$ and $|(z_1)^2 - (z_2)^2| < 1$. $\square$

Therefore we will consider a smaller subset of the moment polytope, P_{sym}, shown in Figure 3.7.

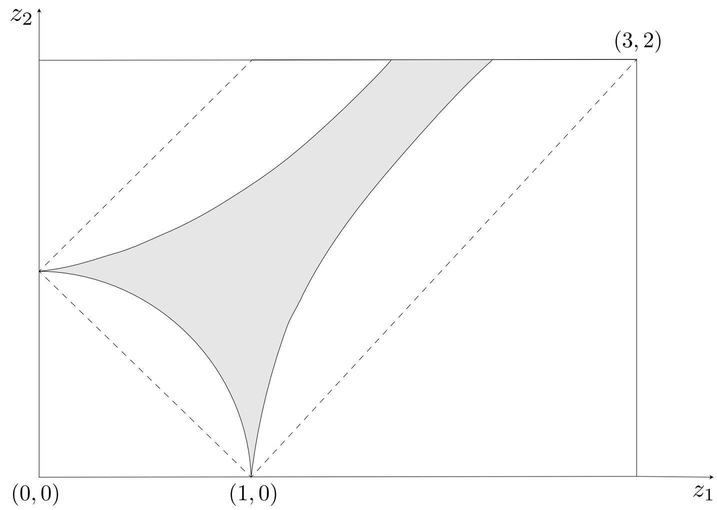


Figure 3.7: The portion of the moment polytope corresponding to the Z triangulation for a symmetric hexagon.

Figure 3.8 shows how $P_{sym} \times [-\pi, \pi]$ splits into parts depending on which $\varphi_i = 0$. In the next section, each of the four regions are described by the possible Joint Chirality-Curl of the knots in that region.

If we choose a point $(z_1, z_2, \phi_3) \in P_{sym} \times [-\pi, \pi]$, this defines two configurations. First we could choose $\phi_1 = \Phi_1^+(z_1, z_2)$, then we get a configuration which is either a symmetric right-handed trefoil or symmetric unknot. As shown in Figure 3.8 either $\varphi_1(z_1, z_2, \phi_3) = 0$ or $\varphi_2(z_1, z_2, \phi_3) = 0$. If $\varphi_1 = 0$, then $\phi_2 = \Phi_2^{++}(z_1, z_2, \phi_3)$ for H to be symmetric. Alternatively, if $\varphi_2 = 0$, then $\phi_2 = \Phi_2^{+-}(z_1, z_2, \phi_3)$ so that H is symmetric. Consider as an example, $(.45, .9, \frac{\pi}{4}) \in P_{sym} \times [-\pi, \pi]$ and $\phi_1 = \Phi_1^+(.45, .9)$. Since $\varphi_1(.45, .9, \frac{\pi}{4}) = 0$, then ϕ_2 must be equal to $\Phi_2^{++}(.45, 9, \frac{\pi}{4})$ in order for the hexagon to be symmetric. Figure 3.9 shows the equilateral symmetric hexagon with action-angle coordinates $(.45, .9, .9, \Phi_1^+(.45, .9), \Phi_2^{++}(.45, .9), \frac{\pi}{4})$. This hexagon has Joint Chirality-Curl $(1, 1)$. We could also choose $\phi_1 = \Phi_1^-(z_1, z_2)$. As seen in Figure 3.8, either $\varphi_3(z_1, z_2, \phi_3) =$

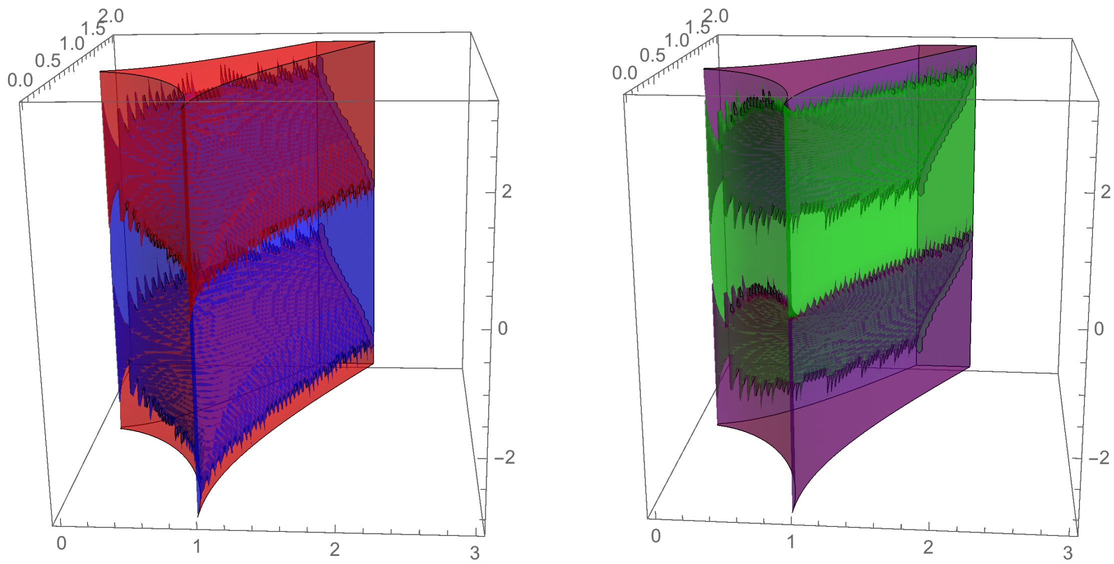


Figure 3.8: The figure on the left shows portions of $P_{sym} \times [-\pi, \pi]$ where $\varphi_1 = 0$ and $\varphi_2 = 0$. This means that H is symmetric with $\phi_1 = \Phi_1^+$ and $\phi_2 = \Phi_2^{++}$ or $\phi_2 = \Phi_2^{+-}$, shown in red and blue, respectively. Similarly, the figure on the right shows portions of $P_{sym} \times [-\pi, \pi]$ where $\varphi_3 = 0$ and $\varphi_4 = 0$. This means that H is symmetric with $\phi_1 = \Phi_1^-$ and $\phi_2 = \Phi_2^{-+}$ or $\phi_2 = \Phi_2^{--}$, shown in purple and green, respectively.

0 or $\varphi_4(z_1, z_2, z_3) = 0$. If $\varphi_3 = 0$, then ϕ_2 must be equal to $\Phi_2^{-+}(z_1, z_2, \phi_3)$ for the hexagon to be symmetric. If $\varphi_4 = 0$, then $\phi_2 = \Phi_2^{--}(z_1, z_2, \phi_3)$. Again consider the example, $(.45, .9, \frac{\pi}{4}) \in P_{sym} \times [-\pi, \pi]$ but now $\phi_1 = \Phi_1^-(.45, .9)$. Since $\varphi_3(.45, .9, \frac{\pi}{4}) = 0$, then ϕ_2 must be equal to $\Phi_2^{-+}(.45, 9, \frac{\pi}{4})$. Figure 3.10 shows the equilateral symmetric hexagon with action-angle coordinates $(.45, .9, .9, \Phi_1^-(.45, .9), \Phi_2^{-+}(.45, .9), \frac{\pi}{4})$. This hexagon has Joint Chirality-Curl $(-1, 1)$.

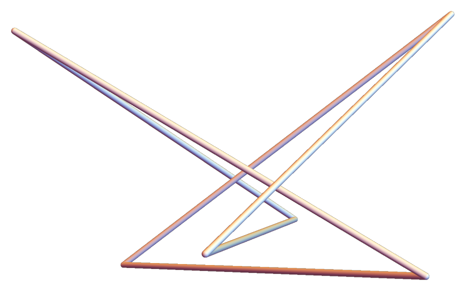


Figure 3.9: This figure shows a symmetric equilateral right-handed hexagonal trefoil with $(z_1, z_2, \phi_3) = (.45, .9, \frac{\pi}{4})$ where ϕ_1 is chosen to be positive.

3.2 Symmetric Equilateral Hexagonal Trefoils

We will now characterize symmetric equilateral hexagons by the Joint Chirality-Curl.

Proposition 3.2.1 *Let H be parametrized with action-angle coordinates using the Z triangulation. If H is a symmetric equilateral hexagon with Joint Chirality-Curl $(1,1)$, then $H = (z_1, z_2, z_2, \Phi_1^{+}(z_1, z_2), \Phi_2^{++}(z_1, z_2, \phi_3), \phi_3)$.*

Proof: Let $H \in Equ_0(6)$. The Z triangulation gives us action-angle coordinates for the space of equilateral hexagons. If H has Joint Chirality-Curl $(1,1)$, then $\Delta_2 = \Delta_4 = \Delta_6 = 1$. We will describe the constraints similar to Proposition 2.2.1. For $\Delta_6 = 1$, then e_2 must piece T_6. For the line through v_1 and v_2 to go through the interior of T_6 the following three equations must be positive:

$$(v_2 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) > 0, \tag{3.5}$$

$$(v_2 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) > 0, \tag{3.6}$$

$$(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0. \tag{3.7}$$

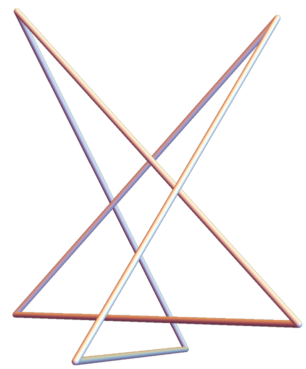


Figure 3.10: This figure shows a symmetric equilateral left-handed hexagonal trefoil with $(z_1, z_2, \phi_3) = (.45, .9, \frac{\pi}{4})$ where ϕ_1 is chosen to be negative.

Suppose H is in standard position for the Z triangulation. Evaluating 3.7 with action-angle coordinates and $z_2 = z_3$, we obtain that

$$\frac{(-(z_1)^4 - (-1 + (z_2)^2)^2 + 2(z_1)^2(1 + (z_2)^2))\sin(\phi_1)}{4z_1} > 0.$$

For $-(z_1)^4 - (-1 + (z_2)^2)^2 + 2(z_1)^2(1 + (z_2)^2) > 0$, then $1 - z_1 < z_2 < 1 + z_1$ for $0 < z_1 < 1$ and $z_1 - 1 < z_2 < 2$ for $1 < z_1 < 3$. These are the triangulation inequalities corresponding to the Z triangulation. Since $(-(z_1)^4 - (-1 + (z_2)^2)^2 + 2(z_1)^2(1 + (z_2)^2))$ is always positive for $(z_1, z_2) \in P_z$, then $\sin(\phi_1) > 0$. Thus $\phi_1 \in (0, \pi)$ and $\phi_1 = \Phi_1^+(z_1, z_2)$ for H to be symmetric.

For $J(H) = (1, 1)$, it is also required that $\Delta_2 = 1$. This means that e_4 must pierce T_2. In order for the line passing through v_4 and v_5 to pierce T_2, the following three inequalities must be satisfied:

$$(v_4 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) > 0, \tag{3.8}$$

$$(v_4 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) > 0, \tag{3.9}$$

$$(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0. \tag{3.10}$$

Again we let H be in standard position for the Z triangulation. Evaluating 3.10 with action-angle coordinates, we obtain that

$$\frac{1}{4}\sqrt{4-(z_2)^2}\sqrt{-(z_1)^4-(-1+(z_2)^2)^2+2(z_1)^2(1+(z_2)^2)}\sin(\phi_2) > 0$$

. Therefore $\sin(\phi_2) > 0$ and so $\phi_2 \in (0, \pi)$. Therefore if H is a symmetric equilateral hexagon with Joint Chirality-Curl $(1, 1)$, then $H = (z_1, z_2, z_2, \Phi_1^+(z_1, z_2), \Phi_2^{++}(z_1, z_2, \phi_3), \phi_3)$ as desired. □

Evaluating the constraints necessary for an equilateral hexagon to have Joint Chirality-Curl $(1, -1)$ and $(-1, \pm 1)$, we can similarly prove the following Propostions.

Proposition 3.2.2 *Let H be parametrized with action-angle coordinates using the Z triangulation. If H is a symmetric hexagon with Joint Chirality-Curl $(1, -1)$, then* $H = (z_1, z_2, z_2, \Phi_1^+(z_1, z_2), \Phi_2^{+-}(z_1, z_2, \phi_3), \phi_3)$.

Proposition 3.2.3 *Let H be parametrized with action-angle coordinates using the Z triangulation. If H is a symmetric hexagon with Joint Chirality-Curl $(-1, 1)$, then* $H = (z_1, z_2, z_2, \Phi_1^-(z_1, z_2), \Phi_2^{-+}(z_1, z_2, \phi_3), \phi_3)$.

Proposition 3.2.4 *Let H be parametrized with action-angle coordinates using the Z triangulation. If H is a symmetric hexagon with Joint Chirality-Curl $(-1, -1)$, then* $H = (z_1, z_2, z_2, \Phi_1^-(z_1, z_2), \Phi_2^{--}(z_1, z_2, \phi_3), \phi_3)$.

Next we prove a constraint on the lengths of the diagonals from the Z triangulation so that we have a symmetric equilateral hexagonal trefoil.

Lemma 3.2.5 *Let $H \in Equ_0(6)$ and parametrize H with action-angle coordinates from the Z triangulation. If H is a symmetric equilateral hexagonal trefoil than $z_1 < 1$.*

Proof: Let $H \in Equ_0(6)$. The Z triangulation of H gives action-angle coordinates to describe the space. Suppose H is symmetric and $z_1 = 1$. Therefore the distance between v_2 and v_5 and the distance between v_3 and v_6 are both equal one. Then the triangles spanned by (v_2, v_5, v_6) and (v_2, v_3, v_6) are isosceles and isometric to the triangle spanned by (v_1, v_2, v_6). All three triangles share a common edge, the segment connecting v_2 to v_6. Let ϕ_1 be fixed. As ϕ_3 varies triangle spanned by (v_1, v_2, v_6) rotates around the segment connecting v_2 and v_6. For some angles of ϕ_3 the triangle spanned by (v_1, v_2, v_6) coincides with the triangle spanned by (v_2, v_5, v_6) and (v_2, v_3, v_6). The edge e_2 can not pierce T_6 and e_5 can not pierce T_2. Therefore H is unknotted. If $z_1 > 1$ then H will also be unknotted. Thus if H is symmetric and knotted then $z_1 < 1$. $\square$

Since $z_1 < 1$, then $z_2 < \sqrt{2}$ for H to be a symmetric equilateral hexagon with Joint Chirality-Curl $(\pm 1, \pm 1)$.

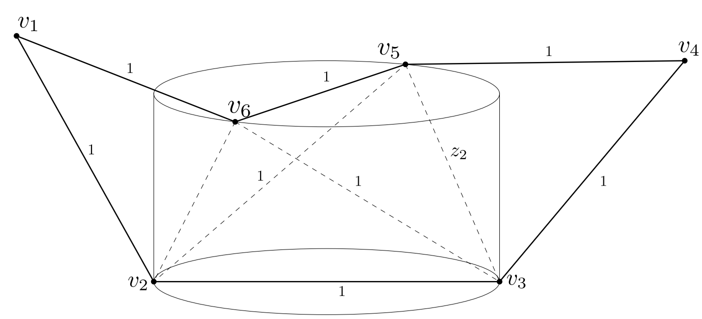


Figure 3.11: If H is symmetric then e_2 and e_5 lie on opposite disks of a cylinder with radius $\frac{1}{2}$. This figure shows the case where $z_1 = 1$ and therefore the triangles spanned by (v_3, v_5, v_6), (v_3, v_5, v_2), and (v_2, v_6, v_5) are isosceles and isometric.

Chapter 4

Polygonal Knots with Thickness

As seen in Chapter 3, the probability that a random equilateral hexagon is knotted is very small. One question is which of these rare equilateral hexagonal trefoils could we construct with physical material. This motivates us to study equilateral hexagons with thickness. In this section, we prove the existence of a maximally thick symmetric equilateral hexagon trefoil.

4.1 Characteristic Qualities of Polygonal Thickness

Before defining the thickness of a polygonal knot, we will discuss the thickness of a smooth knot.

Definition 4.1.1 *For a C^2 smooth knot K and $x \in K$, let $D_r(x)$ be the disk of radius r centered at x lying in the plane normal to the tangent vector at x. Let*

$$R(K) = \sup\{r > 0 : D_r(x) \cap D_r(y) = \emptyset \text{ for all } x \neq y \in K\}.$$

The quantity $R(K)$ is called the thickness radius of K.

For a fixed spatial configuration of a smooth knot K, $R(K)$ is the radius of the thickest tube that can be placed about K without self-intersections. There are two key features of the thickness radius. First, the tube cannot bend two quickly. Second, two distances along the core curve cannot be closer than twice the thickness radius. These observations lead to a second definition of the thickness radius of K, which the polygonal thickness will be modeled after.

Definition 4.1.2 *For a C^2 knot K, let $MinRad(K)$ be the minimum radius of curvature.*

Definition 4.1.3 *For a C^2 knot K with unit tangent map T, the doubly-critical self-distance is the minimum distance between pairs of distinct points on the knot whose chord is perpendicular to the tangent vectors at both of the points. Formally, let*

$$DC(K) = \{(x, y) \in K \times K : T(x) \perp \bar{xy} \perp T(y), x \neq y\},$$

where $\bar{xy}$ is the chord connecting x and y. Define the doubly-critical self-distance by

$$dcsd(K) = \min\{||x - y|| : (x, y) \in DC(K)\},$$

where $||\cdot||$ is the standard $\mathbb{R}^3$ norm.

Lemma 4.1.4 *[15] Suppose K is a C^2 knot. Then*

$$R(K) = \min\{MinRad(K), dcsd(K)/2\}.$$

We now present a definition for the thickness radius of an equilateral polygonal knot in the spirit of Lemma 4.1.4.

Definition 4.1.5 *For any vertex v_i of an n-edge equilateral polygonal knot P, let ϕ_i be* *the turning angle at v_i. That is the angle between vectors $v_{i+1} - v_i$ and $v_i - v_{i-1}$. Let*

$$Rad(v_i) = \frac{1}{2tan(\phi_i/2)}.$$

Define the polygonal radius of curvature of P by

$$MinRad(P) = \min_{i=1,\cdots,n} Rad(v_i).$$

Note the polygonal radius of curvature at a vertex v_i is the radius of a circular arc that is tangent to both edges adjacent to v_i and intersects the edges at their midpoints, as shown in Figure 4.1.

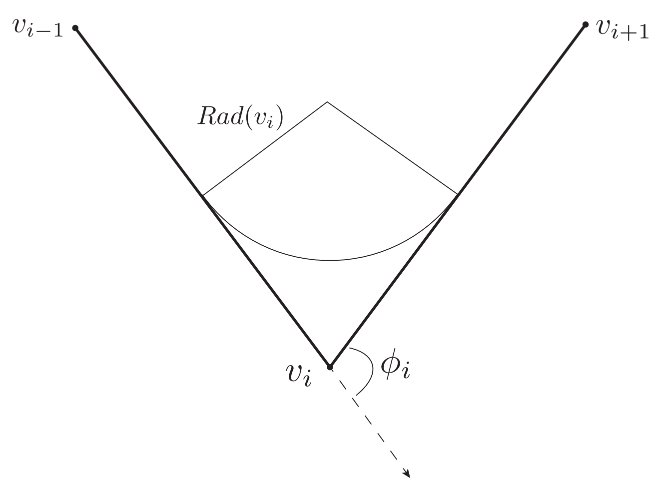


Figure 4.1: This figure shows the polygonal radius of curvature at a vertex v_i. $Rad(v_i)$ is the radius of the circular arc tangent to e_{i-1} and e_i, intersecting both edges at their midpoints.

Next given a point $x \in P$, let $d_x : P \to \mathbb{R}$ be defined by $d_x(y) = ||x - y||$. We call y a turning point for x if d_x changes from increasing to decreasing or decreasing to increasing at y.

Definition 4.1.6 *Let P be an n-edge equilateral polygonal knot. Let the set of doubly-critical pairs be*

$$DC(P) = \{(x, y) \in P \times P : x \neq y, y \textit{ is a turning point for } x, x \textit{ is a turning point for } y\}.$$

Define the doubly-critical self-distance of P by

$$dcsd(P) = \min\{||x - y|| : (x, y) \in DC(P)\}.$$

There are three cases for a pair of points to be doubly-critical. Either both points are on the interiors of edges, both points are vertices, or one point is on the interior of an edge and the other is a vertex.

Definition 4.1.7 *For a equilateral polygonal knot P, let*

$$R(P) = \min\{MinRad(P), \frac{1}{2}dcsd(P)\}.$$

We call $R(P)$ the polygonal thickness radius.

4.2 Equilateral Hexagons with Thickness

In this section, we will focus on how the thickness constraints effect equilateral hexagons. Specifically we will focus on equilateral hexagons that have Joint Chirality-Curl $(1, 1)$, so right-handed trefoils with positive curl. For the polygonal knots there are two features that effect the thickness: the interaction of adjacent edges and the interaction of non-adjacent edges. First we prove that for hexagonal trefoils, it is the interaction of non-adjacent edges that defines the polygonal radius of curvature. This is true since the hexagonal trefoil is realized with a minimal number of edges.

Lemma 4.2.1 *Let $H \in Equ_0(6)$. If H is a trefoil then the polygonal thickness radius of H is $\frac{1}{2}dcsd(H)$.*

Proof: Let $H \in Equ_0(6)$. Suppose towards contradiction that the polygonal thickness radius of H, r, is given by the polygonal radius of curvature at some vertex v_i. Let ϕ_i be the turning angle at v_i. First we will consider the case when the turning angle $\phi_i \leq \frac{\pi}{2}$. Let T_i be the triangle spanned by (v_{i-1}, v_i, v_{i+1}) and P_i be the plane containing T_i. Recall that r is the radius of a circle in P_i that is tangent to both e_{i-1} and e_i and intersects the edges at their midpoints. Let p be the center of this circle. Then p is the circumcenter of T_i, the point at which the perpendicular bisectors of the triangle intersect. If $\phi_i < \frac{\pi}{2}$, then T_i is obtuse. This means that p lies on the exterior of T_i, as shown in Figure 4.2. If $\phi_i = \frac{\pi}{2}$, then T_i is a right triangle and p is the midpoint of the segment connecting v_{i-1} and v_{i+1}.

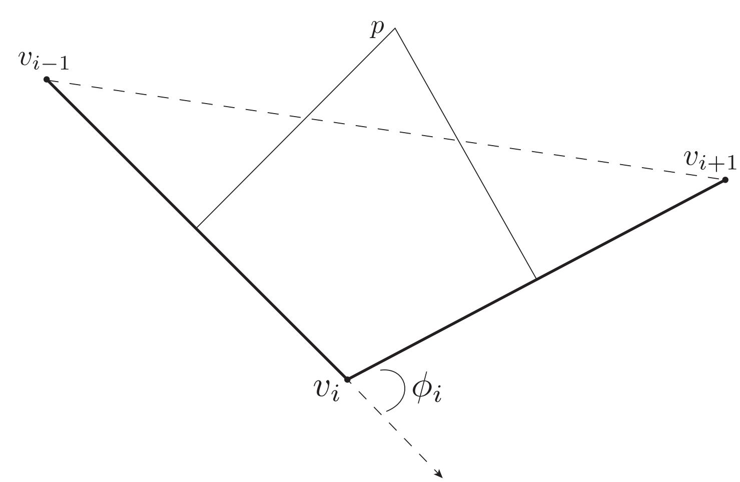


Figure 4.2: When the turning angle is less than $\frac{\pi}{2}$, the triangle spanned by (v_{i-1}, v_i, v_{i+1}) is obtuse. Therefore p is exterior of the triangle.

A cylinder of radius r with axis e_{i-1} intersects the plane P_i in two lines parallel to e_{i-1}, one line passing through p denoted l_{i-1}. Similarly a cylinder of radius r with axis e_i

intersects P_i in two lines parallel to e_i, one line passing through p denoted l_i. Let $x \in P_i$ that is interior of T_i. For any $y \in H$, the distance $d_y(x)$ is less that r, since p is exterior of T_i. So if the polygonal thickness radius is r, the interior of the triangular disk T_i can not be pierced by an edge. Therefore H can not be a trefoil.

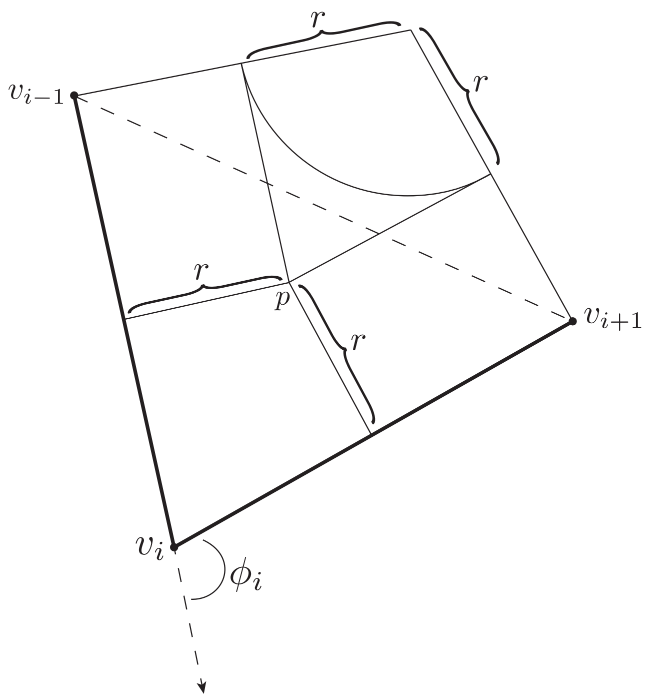


Figure 4.3: When the turning angle at v_i is greater than $\frac{\pi}{2}$, then the triangle spanned by (v_{i-1}, v_i, v_{i+1}) is acute. The circumcenter, p, then lies interior of the triangle.

Next we will consider the case where the turning angle $\phi_i > \frac{\pi}{2}$. Now p, the circumcenter of T_i, lies interior of the triangle, since T_i is acute. Consider again the two lines l_{i-1} and l_i parallel to e_{i-1} and e_i that intersect at p. Suppose that a cylinder of radius r intersects P_i in a circle whose center is distance $2r$ from both v_{i-1} and v_{i+1}. This circle is then tangent to both l_{i-1} and l_i, as shown in Figure 4.3. It is not possible for the center of the circle to intersect T_i. If the polygonal thickness radius is R, then an edge of H can not pierce T_i. Therefore if H is an equilateral hexagonal trefoil, the polygonal thickness radius is defined by the doubly critical self distance. □

Definition 4.2.2 *Let f_{ij} be the minimum distance between non-adjacent edges e_i and e_j.*

For a hexagon, there are nine pairs of non-adjacent edges. Next we will show that for a right-handed equilateral hexagonal trefoil we do not need to consider all nine pairs of non-adjacent edges when computing the polygonal thickness radius.

Lemma 4.2.3 *Let $H \in Equ_0(6)$. Parametrize H with action-angle coordinates corresponding to the T_{135} triangulation. If H is a trefoil with Joint Chirality-Curl $(1,1)$, then the thickness radius of H is $\frac{1}{2}\min\{f_{14}, f_{24}, f_{25}, f_{26}, f_{36}, f_{46}\}$.*

Proof: Let $H \in Equ_0(6)$. First we will prove that $f_{13} \geq f_{14}$. Suppose that $f_{13} = r_1$. Let $x \in e_1$ and $y \in e_3$ such that $(x, y) \in DC(H)$. If $y = v_4$, then $f_{13} \geq f_{14}$ since $y \in e_4$. So we will suppose that y is on the interior of e_3. If x is a non-vertex point of e_1 then e_3 is tangent to a cylinder of radius r_1 with center axis e_1, denoted $Cyl_1(r_1)$, at y. This is shown in Figure 4.4. If $x = v_i$ for $i = 1$ or $i = 2$, then y lies on a sphere of radius r_1 centered at v_i, denote $B_i(r_1)$. In order for H to have Joint Chirality-Curl $(1,1)$, e_4 must interest the interior of T_2, the triangular disk spanned by vertices (v_1, v_2, v_3). Consider the tetrahedron with vertices v_1, v_2, v_3, and y. Since y is a point on e_3, every line through v_4 piercing T_2, the base of the tetrahedron, must intersect the interior of the triangle spanned by (v_1, v_2, y). Since this triangle is in the interior of $Cyl_1(r_1) \cup B_1(r_1) \cup B_2(r_1)$, e_4 passes through this cylinder. Therefore the minimum distance between edges e_1 and e_4 is less that r_1 and $f_{13} \geq f_{14}$, as desired.

For H to have Joint Chirality-Curl $(1,1)$ it is also necessary for e_6 to pierce T_4. Let the minimum distance between e_3 and e_5 be equal to r_2. Similar to the previous argument, if e_6 is to pierce T_4, then the minimum distance between e_6 and e_3 is less than or equal to r_2. Hence $f_{35} \geq f_{36}$. Lastly, e_2 must intersect T_6 for H to have Joint

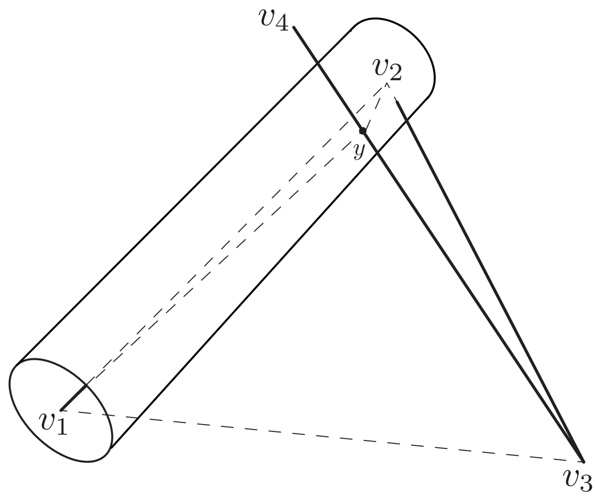


Figure 4.4: This figure shows the cylinder with axis e_1 of radius r_1. The edge e_3 is tangent to the cylinder at some point y, when x is a non-vertex point.

Chirality-Curl $(1,1)$. Let the minimum distance between e_1 and e_5 be equal to r_3. If e_2 pierces T_6, then e_2 passes through the cylinder of radius r_3 with axis e_5. Therefore $f_{15} \geq f_{25}$. Thus, if H has Joint Chirality-Curl, then the polygonal thickness radius of H is $\frac{1}{2}\min\{f_{14}, f_{24}, f_{25}, f_{26}, f_{36}, f_{46}\}$. □

Consider the T_{135} triangulation of H. Recall from Lemma 2.2.2 that all equilateral hexagonal knots with all three diagonals lengths equal are always unknotted. Therefore we analyze the case where all three diagonals of triangulation T_{135} have distinct lengths. Without loss of generality, we assume that $d_2 > d_1, d_3$. With this assumption, we prove the following lemma.

Lemma 4.2.4 *Let $H \in Equ_0(6)$. Parametrize H with action-angle coordinates corresponding to the T_{135} triangulation. Suppose that $d_2 > d_1, d_3$. If H is a right-handed trefoil, then the thickness radius of H is $\frac{1}{2}\min\{f_{14}, f_{24}, f_{25}, f_{26}, f_{36}\}$.*

Proof: Let $H \in Equ_0(6)$. Parametrize H with action-angle coordinates corresponding to the T_{135} triangulation, so $H = (d_1, d_2, d_3, \theta_1, \theta_2, \theta_3)$ and is in standard position. From the previous Lemma, it remains to show that $f_{46} \neq \min\{f_{14}, f_{24}, f_{25}, f_{26}, f_{36}\}$. To this end, we will show that if $d_2 > d_3 > d_1$ and H has Joint Chirality-Curl $(1,1)$, then $f_{46} > f_{36}$. Since $d_2 > d_1, d_3$ then for H to be a $(1,1)$ trefoil, $\theta_2 \in (0, \pi)$ and $\theta_1, \theta_3 \in (0, \frac{\pi}{2})$ by Lemma 2.2.5. Consider the triangle spanned by (v_1, v_3, v_5). Let l_2 be the perpendicular bisector to the segment connecting v_1 and v_3, intersecting at the midpoint, m_2. Similarly, we define l_4 and l_6 to be the perpendicular bisectors to the segments connecting v_3 to v_5 and v_5 to v_1, respectively. The three lines intersect in a unique point, k, the circumcenter of the triangle spanned by (v_1, v_3, v_5). Moreover, l_i represents the projection of v_i onto the xy-plane as θ_i varies and k is the projection onto the xy-plane when v_i coincide for $i = 1, 2, 3$, if such point exists. Note if v_2 and v_4 project onto the same point in the xy-plane they are equal since they are both distance one from v_3. Since $d_3 > d_1$, then l_4 intersects the segment connecting v_1 and v_5 instead of the segment connecting v_1 and v_3. If the projection onto the xy-plane of v_2 lies on the segment connecting m_2 and k then the plane perpendicular to the xy-plane containing l_2 separates e_4 and T_2. Therefore H can not have Joint Chirality-Curl $(1,1)$. This defines a range of allowable θ_1. If the projection of v_6 onto the xy-plane lies on the segment connecting m_6 and k, then the plane perpendicular to the xy-plane containing l_2 separates e_2 and T_6. Thus H can not be a $(1,1)$ trefoil. Next we will show that for H to have Joint Chirality-Curl $(1,1)$, the projection of v_4 onto the xy-plane will not lie on the segment connecting m_4 and k. Let p be the point where e_1 intersects e_4 when $\theta_1 = 0$ and $\theta_2 = \pi$. In order for e_4 to intersect T_2, e_4 must intersect the cone spanned by e_1. The two cones will intersect along an arc connecting p to point which projects onto k. Let ψ be the angle for θ_2 for which the orthogonal projection of v_4 onto the xy-plane is k. If $\theta_2 < \psi$, then e_4 no longer intersects the cone spanned by e_1. Therefore the projection of v_4 onto the xy-plane is not on the segment connecting m_4 and k for H to have Joint Chirality-Curl $(1,1)$.

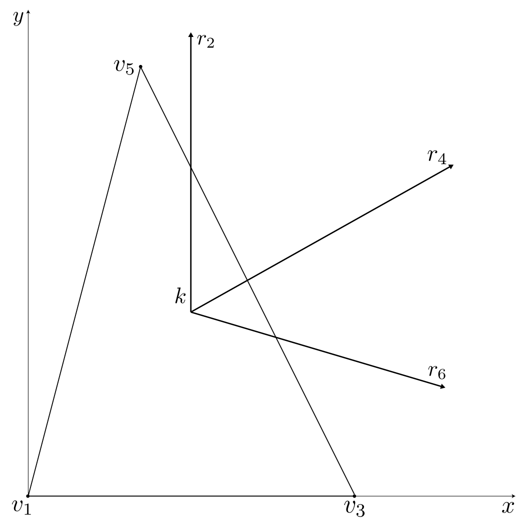


Figure 4.5: This figure show rays r_i denoting where θ_i will project if H has Joint Chirality-Curl $(1,1)$.

Let r_i be the ray starting at k along l_i, as shown in Figure 4.5. Then we have shown that for H to be Joint Chirality-Curl $(1,1)$, v_i must project onto r_i for all $i = 1,2,3$. Define P_4 to be the plane containing the triangle spanned by (v_3, v_4, v_5). Let P_a be the plane perpendicular to P_4 containing the altitude of T_4. Given any allowable angles so that v_i project on r_i, P_a separates e_6 and e_4. Suppose that $f_{46} = m$ and define $Cyl_4(m)$ to be the cylinder of radius m with axis e_4. Then e_6 will be tangent to $Cyl_4(m)$ at some point p. Note that P_a intersects $Cyl_4(m)$, separating it into two cylinders $Cyl_4^+(m)$ and $Cyl_4^-(m)$. Let $Cyl_4^+(m)$ be the part containing v_5, as shown in Figure 4.6.

Then p must be on Cyl_4^- since P_a separates e_4 and e_6. Let Cyl_3 be the cylinder of radius m with core e_3. Then Cyl_4^- lies interior of Cyl_3. Therefore $p \in e_6$ is interior of Cyl_3 and $f_{36} < m$. Hence for H to have Joint Chirality-Curl $(1,1)$ with $d_2 > d_3 > d_1$, the polygonal radius of curvature is $\frac{1}{2}\min\{f_{14}, f_{24}, f_{25}, f_{26}, f_{36}\}$. $\square$

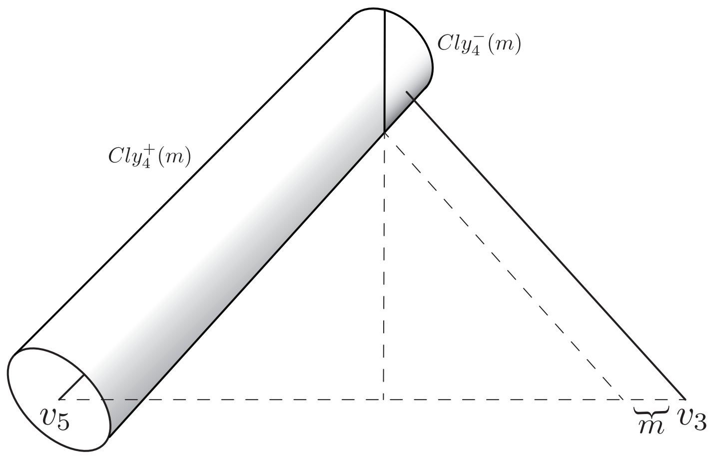


Figure 4.6: This figure shows the cylinder about e_4 of radius m. The cylinder is split into two parts by the plane perpendicular to P_4 through the altitude of the triangle spanned by (v_3, v_4, v_5).

4.3 Maximally Thick Equilateral Hexagons

In this section, we discuss the maximum thickness for an equilateral hexagonal trefoil. First we use computer simulations to discover local maximums of the polygonal thickness radius. We start with an equilateral hexagon H with Joint Chirality-Curl $(1, 1)$. Next we perform random perturbations of the action-angle coordinates, choosing the greater thickness conformation at each stage. Recall that the polygonal thickness radius for a hexagonal trefoil is determined by the minimum distance of non-adjacent edges. If we deform the knot in a way that changes the Joint Chirality-Curl, then would pass through a singular conformation. By choosing the conformation with larger thickness at each stage and keeping the perturbations small, we ensure that the polygons generated have Joint Chirality-Curl $(1, 1)$. Thus we obtain a sequence of polygons $H_0, H_1, H_2, \ldots$ such that $J(H_i) = (1, 1)$ and the thickness of H_i increases monotonically towards a limiting conformation.

In the numerical experiments, the limiting conformation appeared to have symmetry. Optimizing within the space of symmetric hexagonal trefoils led to a slightly better maximum. We verify that this conformation, found numerically, is a maximally thick symmetric equilateral hexagon with Joint Chirality-Curl $(1,1)$.

Theorem 4.3.1 *There exists a locally maximally thick symmetric equilateral hexagonal trefoil.*

Proof: Consider the component on $Equ(6)$ consisting of hexagons with Joint Chirality-Curl $(1,1)$. Let $H \in Equ_0(6)$ and use the Z triangulation to define action-angle coordinates for the space. Suppose that H is symmetric and translate and rotate H so that v_2 is the origin, v_5 is on the positive x-axis, and v_6 is on the xy-plane with v_2, v_5, and v_6 oriented in a counter-clockwise direction. By Proposition 3.2.3, H is determined by $(z_1, z_2, \phi_3) \in P_{sym} \times (0, \pi)$ and $H = (z_1, z_2, z_2, \Phi_1^+(z_1, z_2), \Phi_2^{++}(z_1, z_2, \phi_3), \phi_3)$. Recall that f_{ij} is the minimum distance between edges e_i and e_j. Let p_i be a point on e_i, $d_{ij} = p_j - p_i$, and $u_i = v_{i+1} - v_i$, where v_i are defined with action-angle coordinates as in Chapter 4. If the minimum distance between two edges occurs at a point on the interior of the edge then f_{ij} are defined as

$$\begin{aligned}
f_{14}(z_1, z_2, \phi_3) &= \frac{|d_{14} \cdot (u_1 \times u_4)|}{||u_1 \times u_4||}, \\
f_{24}(z_1, z_2, \phi_3) &= \frac{|d_{24} \cdot (u_2 \times u_4)|}{||u_2 \times u_4||}, \\
f_{26}(z_1, z_2, \phi_3) &= \frac{|d_{26} \cdot (u_2 \times u_6)|}{||u_2 \times u_6||}, \\
f_{25}(z_1, z_2, \phi_3) &= \frac{|d_{25} \cdot (u_2 \times u_5)|}{||u_2 \times u_5||}, \\
f_{36}(z_1, z_2, \phi_3) &= \frac{|d_{36} \cdot (u_3 \times u_6)|}{||u_3 \times u_6||}, \\
f_{46}(z_1, z_2, \phi_3) &= \frac{|d_{46} \cdot (u_4 \times u_6)|}{||u_4 \times u_6||}.
\end{aligned}$$

Define $f(z_1, z_2, \phi_3) = \min\{f_{14}, f_{24}, f_{26}, f_{36}, f_{25}, f_{46}\}$. If H has Joint Chirality-Curl $(1,1)$ and is symmetric, then the polygonal thickness radius of H is equal to $\frac{1}{2}f(z_1, z_2, \phi_3)$. Additionally if H is symmetric, then $f_{14} = f_{36}$ and $f_{24} = f_{26}$, as demonstrated in Figure 4.7. Therefore it suffices to consider f_{14}, f_{24}, f_{25}, and f_{46}.

Let $p = (0.466079, 0.916184, 0.875994) \in P_{sym} \times (0, \pi)$ and consider $f(p) = 5.613739 \times 10^{-2}$. Evaluating each minimum distance function individually at p, we see that $f_{14}(p) = f_{24}(p) = 5.613739 \times 10^{-2}$. In order to increase the thickness, we must then increase both functions. However, the tangent planes to f_{14} and f_{24} at p coincide. Additionally, the gradients of f_{14} and f_{24} at p point in opposite directions, specifically

$$\frac{\nabla f_{14}(p)}{||\nabla f_{14}(p)||} = \{0.327675, 0.941483, -0.0789877\}$$

and

$$\frac{\nabla f_{24}(p)}{||\nabla f_{24}(p)||} = \{-0.327675, -0.941483, 0.0789877\}.$$

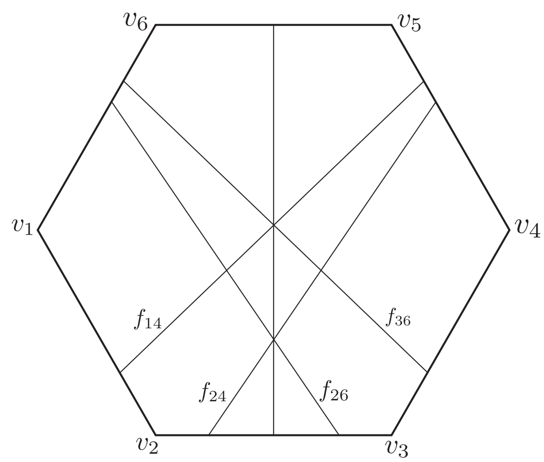


Figure 4.7: In this figure doubly-critical pairs of H are connected in the planar equilateral hexagon.

Therefore it is impossible to simultaneously increase f_{14} and f_{24} and f has a local maximum at p. $\square$

Recall the automorphism s that acts on $Equ(6)$ by shifting the order of the vertices. By Theorem 1.1.15, s^2 does not change the Joint Chirality-Curl of an equilateral hexagon. Consider the action-angle coordinates from the T_{135} triangulation for the locally maximally thick symmetric configuration from Theorem 4.3.1. If $a = 0.609323$, $b = 0.916184$, $c = 0.812150$, $\alpha = 0.479581$, $\beta = 1.755246$, and $\gamma = 0.251224$, then $H_1 = (a, b, c, \alpha, \beta, \gamma)$ defines the configuration with action-angle coordinates from the T_{135} triangulation. The diagonals with lengths d_1, d_2, and d_3 from the T_{135} triangulation connect the odd vertices. When $d_1 = d_2 = d_3$ the hexagon is always geometrically equivalent to the unknot. So we consider the generic case where the three diagonal lengths are distinct. Shifting the vertices of H_1 twice, we get $s^2(H_1) = H_2 = (b, c, a, \beta, \gamma, \alpha)$. Shifting the vertices of H_2 twice, we arrive at $s^2(H_2) = H_3 = (c, a, b, \gamma, \alpha, \beta)$. For H_1, H_2, H_3 the largest diagonal length is d_2, d_1, and d_3, respectively. So there are three configurations with Joint Chirality-Curl $(1, 1)$ with equal polygonal thickness radius, shown in Figures 4.8, 4.9, 4.10.

However, only H_1 is symmetric, that is invariant under the reflection r_1. Due to the shifting of the vertices, H_2 and H_3 are invariant under the reflections r_3 and r_2, respectively. Taking into consideration the configurations with Joint Chirality-Curl $(-1, 1)$, $H_4 = (c, b, a, \gamma, \beta, \alpha)$, $H_5 = (b, a, c, \beta, \alpha, \gamma)$, $H_6 = (a, c, b, \alpha, \gamma, \beta)$, and the corresponding configurations with negative curl, we get twelve equilateral hexagonal trefoils with polygonal thickness radius 2.806869×10^{-2}. Only four are symmetric, one for each type of hexagonal trefoil.

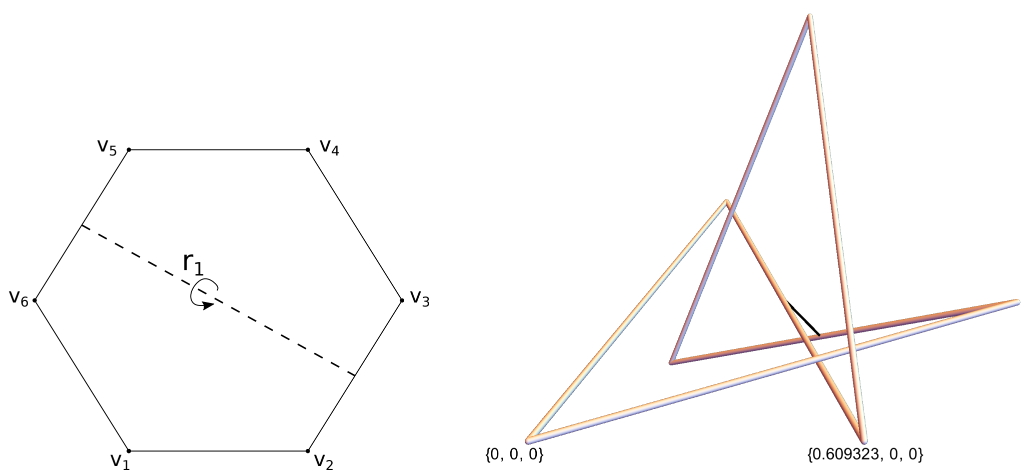


Figure 4.8: This figure shows equilateral hexagon H_1 with action-angle coordinates from the T_{135} triangulation $(a, b, c, \alpha, \beta, \gamma)$. Vertices v_1 and v_3 are labeled, for orientation. In addition, the segment connecting the midpoint of e_2 and e_5 is shown. This segment represents the minimum distance between edges e_2 and e_5 and is the axis of rotation for the symmetry.

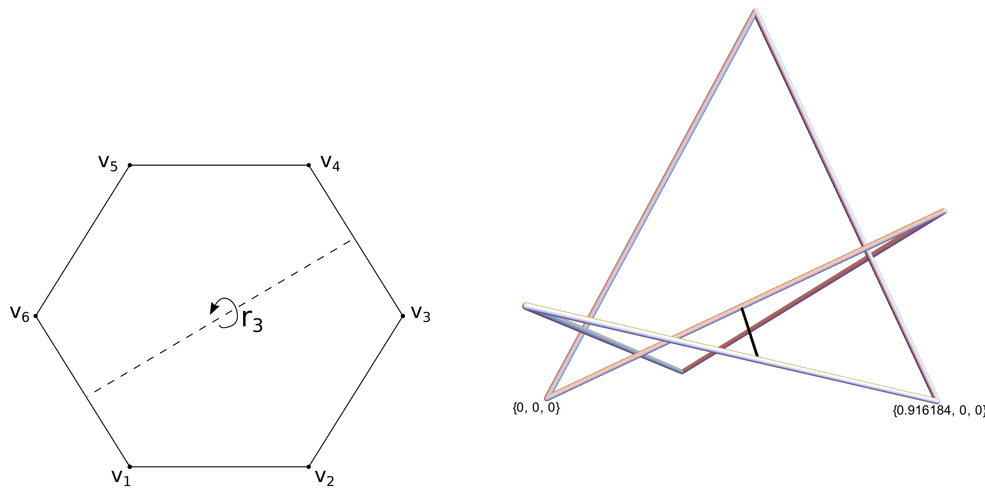


Figure 4.9: This figure shows equilateral hexagon H_2 with action-angle coordinates from the T_{135} triangulation $(b, c, a, \beta, \gamma, \alpha)$. Vertices v_1 and v_3 have been labeled. In addition, the segment connecting the midpoint of e_3 and e_6 is shown.

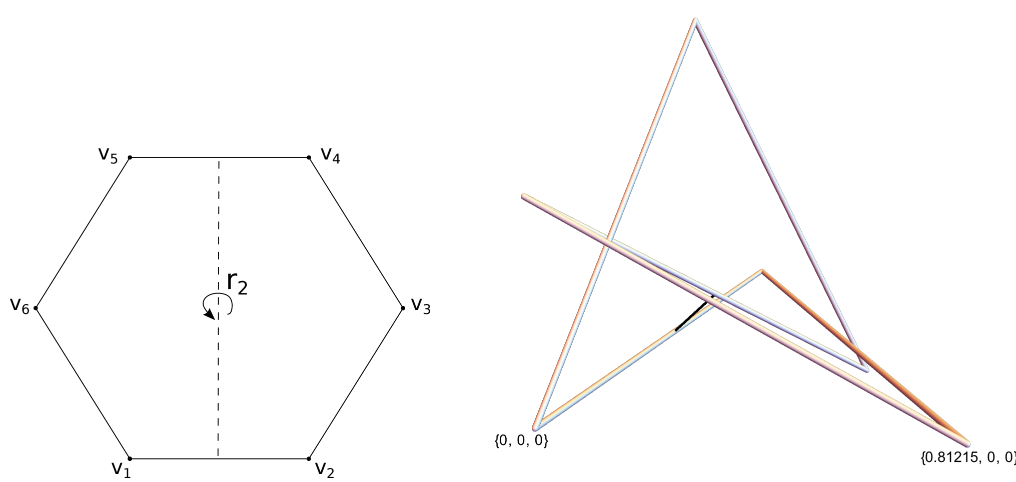


Figure 4.10: This figure shows equilateral hexagon H_3 with action-angle coordinates from the T_{135} triangulation $(c, a, b, \gamma, \alpha, \beta)$. Similar to the previous figures, vertices v_1 and v_3 are labeled. The segment connecting the midpoint of e_1 and e_4 is shown.

Next we will show that the polygonal thickness radius of the configuration with symmetric coordinates $p = (z_1, z_2, \phi_3) = (0.466079, 0.916184, 0.875994)$ is the highest thickness value for a symmetric hexagon.

Theorem 4.3.2 *Let $H \in Equ_0(6)$. If H is a symmetric trefoil, then 2.806869×10^{-2} is a lower bound for the maximum polygonal thickness radius. In particular, if $J(H) = (1,1)$ and H is parametrized with action-angle coordinates from the Z triangulation, this lower bound is obtained when*

$$H = (0.466079, 0.916184, 0.916184, 0.344998, 0.148769, 0.875994).$$

Proof: Let $H \in Equ_0(6)$. It suffices to consider symmetric hexagons with Joint Chirality-Curl $(1,1)$. If H has Joint Chirality-Curl $(1,1)$ then triangular disks T_2, T_4, and T_6 are pierced by edges e_4, e_6, and e_2, in that order. As described in Proposition 2.2.1, there are 12 inequalities that must be satisfied so that these intersections occur:

$$(v_6 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) > 0,$$

$$(v_6 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) > 0,$$

$$(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) > 0,$$

$$-\big((v_6 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big)\big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) > 0,$$

$$(v_2 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) > 0,$$

$$(v_2 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) > 0,$$

$$(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0,$$

$$-\big((v_2 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big)\big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) > 0,$$

$$(v_4 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) > 0,$$

$$(v_4 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) > 0,$$

$$(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0,$$

$$-\big((v_4 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big)\big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) > 0.$$

We will translate H so that v_2 is the origin. Additionally, rotate H so that v_5 is on the positive x-axis and v_6 is on the upper xy-plane. Then evaluating these conditions with action-angle coordinates from the Z triangulation gives twelve functions, f_i, g_i, h_i, j_i for $i = 1, 2, 3$ that must be positive so that H has Joint Chirality-Curl $(1, 1)$. Further since H is symmetric, we let $z_3 = z_2$, $\phi_1 = \Phi_1^+(z_1, z_2)$, and $\phi_2 = \Phi_2^{++}(z_1, z_2, \phi_3)$ and define f_i, g_i, h_i, j_i in terms of z_1, z_2, ϕ_3. Therefore we can define the symmetric trefoil region to be the subset of $P_{sym} \times [0, \pi]$ where $f_i > 0, g_i > 0, h_i > 0$ and $j_i > 0$. In Theorem 4.3.1, we numerically verify that the configuration with $p = (z_1, z_2, \phi_3) = (0.466079, 0.916184, 0.875994)$ was locally a maximum for the polygonal thickness radius for symmetric equilateral hexagonal trefoils. Recall for a symmetric equilateral hexagonal trefoil with Joint Chirality-Curl $(1, 1)$ it suffices to consider f_{14}, f_{24}, f_{25}, and f_{46}, where f_{ij} is the minimum distance between non-adjacent edges e_i and e_j. Fixing $\phi_3 \in [0, \pi]$ we can take slices of the symmetric trefoil region and plot the contours for f_{14}, f_{24}, f_{25}, and f_{46}. The contour analysis verifies that p is an absolute maximum of the thickness for symmetric equilateral hexagons with Joint Chirality-Curl $(1, 1)$. Therefore 2.806869×10^{-2} is a lower bound for the maximum polygonal thickness radius. $\square$

Let Γ be the path-component of $Equ(6)$ consisting of equilateral hexagons with Joint Chirality-Curl $(1, 1)$. If $H \in \Gamma$, then the diagonal lengths d_i from the T_{135} triangulation are not equal. Consider a generic configuration where all diagonal lengths are distinct.

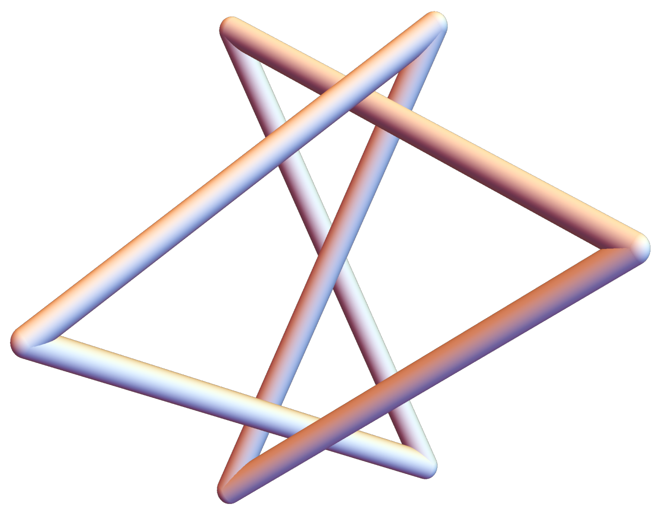


Figure 4.11: Symmetric equilateral hexagonal trefoil with maximal thickness.

As discussed earlier, shifting the vertices of H twice or cyclically permuting the d_i's and θ_i's yields a configuration with Joint Chirality-Curl $(1, 1)$ and equal polygonal thickness radius, R. So we will consider the subset of Γ where d_2 is the largest diagonal length, denoted Γ_2. Suppose that P is a unique point in Γ_2 such that $R(P) = \sup_{H\in\Gamma_2} R(H)$. Through translations and rotations we may assume that v_1, v_3, and v_5 are on the xy-plane oriented in a counter-clockwise orientation. Let $(a, b, c, \alpha, \beta, \gamma)$ be the action-angle coordinates of P. Since $J(P) = (1, 1)$ then v_2, v_4, and v_6 have positive z-coordinates and span a triangle above the xy-plane. Let $a' = ||v_2 - v_4||$, $b' = ||v_6 - v_2||$, and $c' = ||v_4 - v_6||$. Additionally let α', β', γ' be the corresponding angles around diagonals connecting the vertices of the triangle spanned by (v_2, v_4, v_6). Define P' to be the configuration with action-angle coordinates in the T_{135} triangulation $(a', b', c', \alpha', \beta', \gamma')$. The two spatial configurations P and P' are equivalent up to translations and rotations and so they have the same polygonal thickness radius. If $b' >, a', c'$, then since P was a unique extremal point for Γ_2 then $a = a'$, $b = b'$, $c = c'$, $\alpha = \alpha'$, $\beta = \beta'$, and $\gamma = \gamma'$. By definition, this means that P is symmetric. Therefore we conjecture that 2.806869×10^{-2} is also the maximal polygonal thickness radius for all equilateral hexagonal trefoils. Numerical evidence, discussed in Chapter 5, supports this conjecture.

Chapter 5

Numerical Efforts

5.1 Visual Tools

A major of benefit of looking at the space of equilateral hexagons, up to translations and rotations, is that the space can be parametrized over the product of two 3-dimensional spaces. Using the constraints of Proposition 2.2.1, we can fix a point in the moment polytope and use Mathematica's RegionPlot3D to show the portion of T^3, represented as the cube $[0, 2\pi]^3$, corresponding to trefoils with Joint Chirality-Curl $(\pm 1, \pm 1)$. This not only gives a sense of what the region looks like but it also allows a way to easily sample equilateral hexagonal trefoils. Figure 5.1 shows the possible angles for the four different types of equilateral hexagonal trefoils with diagonal lengths coming from the T_{135} triangulation fixed at $d_1 = 0.8$, $d_2 = 0.7$, and $d_3 = 0.6$. Equilateral hexagonal trefoils with Joint Chirality-Curl $(1, 1)$ and $(-1, 1)$ have angles between 0 and π and are shown in yellow and red, respectively. Equilateral hexagonal trefoils with Joint Chirality-Curl $(1, -1)$ and $(-1, -1)$ have angles between π and 2π and are shown in green and blue, respectively.

In Chapter 3, symmetric configurations are studied. The Z triangulation gives coordi-

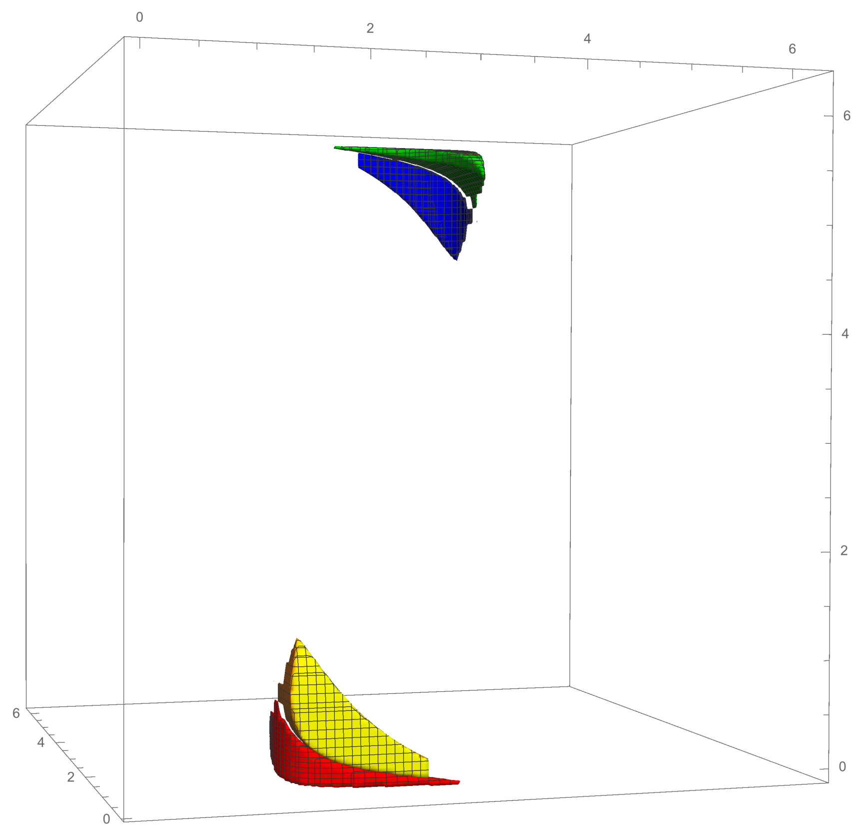


Figure 5.1: A point $(0.8, 0.7, 0.6)$ is fixed in the moment polytope corresponding to the T_{135} triangulation. This figure shows angles $(\theta_1, \theta_2, \theta_3) \in [0, 2\pi]^3$ from the T_{135} triangulation that correspond to equilateral hexagonal trefoils with Joint Chirality-Curl $(\pm 1, \pm 1)$.

nates $(z_1, z_2, z_3, \phi_1, \phi_2, \phi_3)$ for the space of equilateral hexagons. There are four equations that need to be satisfied

$$||v_3 - v_1|| - ||v_4 - v_2|| = 0, \tag{5.1}$$

$$||v_5 - v_3|| - ||v_6 - v_2|| = 0, \tag{5.2}$$

$$||v_5 - v_1|| - ||v_6 - v_4|| = 0, \tag{5.3}$$

$$||v_5 - v_2|| - ||v_6 - v_3|| = 0. \tag{5.4}$$

Using 5.1, 5.2, 5.4 we are able to solve for z_3, ϕ_1, and ϕ_2 in terms of z_1, z_2, and ϕ_3. Lastly, Equation 5.3 must be satisfied. We use Mathematica's graphical tools to verify that there is a symmetric equilateral hexagon, possibly singular, for all $\phi_3 \in [-\pi, \pi]$. Additionally, we can plot the region of symmetric equilateral hexagonal trefoils in the space of symmetric equilateral hexagons. As shown previously, the subset of equilateral hexagons with Joint Chirality-Curl $(1, 1)$ are the configurations $(v_1, v_2, \ldots, v_6)$ whose vertices satisfy the following inequalities:

$$(v_6 - v_1) \times (v_4 - v_1) \cdot (v_3 - v_1) > 0,$$

$$(v_6 - v_1) \times (v_5 - v_1) \cdot (v_4 - v_1) > 0,$$

$$(v_6 - v_1) \times (v_3 - v_1) \cdot (v_5 - v_1) > 0,$$

$$-\big((v_6 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big)\big((v_1 - v_3) \times (v_5 - v_3) \cdot (v_4 - v_3)\big) > 0,$$

$$(v_2 - v_3) \times (v_6 - v_3) \cdot (v_5 - v_3) > 0,$$

$$(v_2 - v_3) \times (v_1 - v_3) \cdot (v_6 - v_3) > 0,$$

$$(v_2 - v_3) \times (v_5 - v_3) \cdot (v_1 - v_3) > 0,$$

$$-\big((v_2 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big)\big((v_3 - v_5) \times (v_1 - v_5) \cdot (v_6 - v_5)\big) > 0,$$

$$(v_4 - v_5) \times (v_2 - v_5) \cdot (v_1 - v_5) > 0,$$

$$(v_4 - v_5) \times (v_3 - v_5) \cdot (v_2 - v_5) > 0,$$

$$(v_4 - v_5) \times (v_1 - v_5) \cdot (v_3 - v_5) > 0,$$

$$-\big((v_4 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big)\big((v_5 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1)\big) > 0.$$

Next we parametrize $Equ_0(6)$ with action-angle coordinates from the Z triangulation. Evaluating these inequalities with $z_2 = z_3$, $\phi_1 = \Phi_1^+(z_1, z_2)$, and $\phi_2 = \Phi_2^{++}(z_1, z_2, \phi_3)$, we can then plot the region of $P_{sym} \times [-\pi, \pi]$ where the inequalities are satisfied. We can similarly plot the regions of symmetric equilateral hexagons with Joint Chirality-Curl $(1, -1)$, $(-1, 1)$, and $(-1, -1)$.

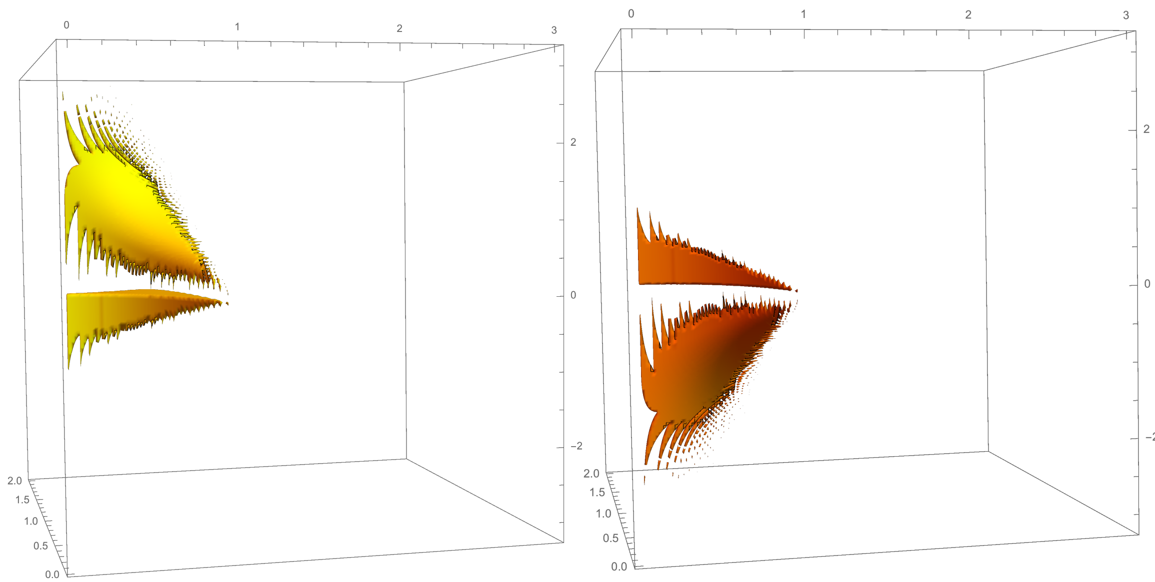


Figure 5.2: The figure on the left shows the region of $[0, 3] \times [0, 2] \times [-\pi, \pi]$ corresponding to symmetric equilateral hexagons with Joint Chirality-Curl $(1, 1)$ and $(1, -1)$. The figure on the right shows the region of $[0, 3] \times [0, 2] \times [-\pi, \pi]$ corresponding to symmetric equilateral hexagons with Joint Chirality-Curl $(-1, 1)$ and $(-1, -1)$.

5.2 Computational Results

We use computer simulations to search for local maximums of the polygonal thickness radius. First the T_{135} triangulation give action-angle coordinates on the space of equilateral hexagons. We start with an equilateral hexagon H with Joint Chirality-Curl $(1, 1)$ and calculate the polygonal thickness radius. Next we perform small perturbations of each action-angle variable independently by a randomly value chosen between a small interval $[-\delta, \delta]$. The polygon with greater thickness is chosen and the process repeated. By improving the thickness at each iteration and keeping the perturbations small, we ensure that the polygons generated have Joint Chirality-Curl $(1, 1)$. Thus we obtain a sequence of conformations $\{H_i\}$ with $J(H_i) = (1, 1)$ such that the thickness of H_i increases monotonically. Computations are done in Mathematica. Initially, δ is chosen to be 10^{-6} and the thickness increases very quickly with the random perturbation method. Eventually the process slows down, and there are no improvements in thickness. We continue to run random perturbations with smaller δ until there are no improvements in thickness.

In Chapter 4, we prove that the configuration with action-angle coordinates from the Z triangulation $(0.466079, 0.916184, 0.916184, 0.344998, 0.148769, 0.875994)$ achieves the maximum polygonal thickness radius for symmetric equilateral hexagonal trefoils. We conjecture that this is in fact the maximum polygonal thickness radius for all equilateral hexagonal trefoils. Numerical evidence supports that this configuration locally maximizes the polygonal thickness radius. The desired accuracy of thickness and precision capabilities of Mathematica become important concerns. In initial computations, the standard working precision of Mathematica is used and so the minimum δ value is taken to be 10^{-16}. Globally increasing the working precision, we test $\delta = 10^{-50}$ and still no improvements of the thickness with over 100 million random perturbations. This supports the conjecture that we have a local maximum of the thickness function over the space of all equilateral hexagonal trefoils.

In addition to searching for maximums of the thickness function, we also use Mathematica to estimate the knotting probability of equilateral hexagons. In Chapter 2, we described the Monte Carlo method for testing whether a random equilateral hexagon had Joint Chirality-Curl $(1, 1)$. We estimate that the knot probability for equilateral hexagons is about 1.370402×10^{-4}. We run a similar experiment but for symmetric equilateral hexagons. We randomly sample a symmetric configuration and test whether the configuration is a trefoil. We estimate that the ratio of symmetric equilateral hexagonal trefoils to symmetric equilateral hexagons is 1.788784×10^{-3}.

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